Alexander S. Kravchuk
Pekka J. Neittaanmäki

Solid Mechanics and Its Applications

Variational and Quasi-Variational Inequalities in Mechanics

Springer

VARIATIONAL AND QUASI-VARIATIONAL INEQUALITIES IN MECHANICS

SOLID MECHANICS AND ITS APPLICATIONS
Volume 147

Aims and Scope of the Series

The fundamental questions arising in mechanics are: *Why?, How?,* and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies: vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

For a list of related mechanics titles, see final pages.

Variational and Quasi-Variational Inequalities in Mechanics

by

ALEXANDER S. KRAVCHUK

Moscow State University, Moscow, Russia

and

PEKKA J. NEITTAANMÄKI

University of Jyväskylä, Jyväskylä, Finland

Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-6376-3 (HB)
ISBN 978-1-4020-6377-0 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Printed on acid-free paper

Preface

The variational method is a powerful tool to investigate states and processes in technical devices, nature, living organisms, systems, and economics. The power of the variational method consists in the fact that many of its statements are physical or natural laws themselves.

The essence of the variational approach for the solution of problems relating to the determination of the real state of systems or processes consists in the comparison of close states. The selection criteria for the actual states must be such that all the equations and conditions of the mathematical model are satisfied.

Historically, the first variational theory was the Lagrange theory created to investigate the equilibrium of finite-dimensional mechanical systems under holonomic bilateral constraints (bonds). The selection criterion proposed by Lagrange is the admissible displacement principle. In accordance with this principle, the work of the prescribed forces (supposed to be constant) on infinitesimally small, kinematically admissible (virtual) displacements is zero. It is known that equating the virtual work performed for potential systems to zero is equivalent to the stationarity conditions for the total energy of the system.

The transition from bilateral constraints to unilateral ones was performed by O. L. Fourier. Fourier demonstrated that the virtual work on small disturbances of a stable equilibrium state of a mechanical system under unilateral constraints must be positive (or, at least, nonnegative). Therefore, for such a system the corresponding mathematical model is reduced to an inequality and the problem becomes nonlinear.

The dynamic theory of systems under unilateral constraints was proposed by M. V. Ostrogradski and completed by J. R. Mayer and E. Zermelo. The Ostrogradski method is an algorithm for the integration of the equations of motion. According to this algorithm, only the bonds which reduce to equality must be taken into account, because the strict inequality constraints do not influence the motion. The selection of such bonds is performed using a special method in our work as well.

Later, this approach was generalized to continuous systems, i.e., as a problem of continuum mechanics. At first such a problem was considered by Ostrogradski. The first research into the unilateral problem in the mechanics of solids was performed by A. Signorini in 1936. The Signorini problem consists of finding the equilibrium of a deformed solid in a smooth rigid shell.

The aim of this book is to consider a wide set of problems arising in the mathematical modeling of mechanical systems under unilateral constraints. Most attention is devoted to the interaction of deformed solids. In these investigations elastic and nonelastic deformations and friction and adhesion phenomena are taken into account. All the necessary mathematical tools are given: local boundary value problem formulations, construction of variational equations and inequalities, the transition to minimization problems, existence and uniqueness theorems, and variational transformations (Friedrichs and Young–Fenchel–Moreau) to dual and saddle-point search problems.

Important new results concern contact problems with friction. The Coulomb friction law (and some others), in which relative sliding velocities appear, is considered. The corresponding quasi-variational inequality is constructed as well as an appropriate iterative method for its solution. Convergence is demonstrated. Outlines of the variational approach to nonstationary and dissipative systems and to the construction of the governing equation are also given.

Examples of analytical and numerical solutions are presented. Numerical solutions were obtained with the finite element and boundary element methods (BEMs) with all the necessary definitions and theorems.

For the variational principles and variational methods of the mechanics, the deformed solids are classical tools for the mathematical modeling of processes in technical devices, constructions, and nature. Much has been published on the subject. The foundation of the methods developed in the first half of the past century can be found in [CH53]. More recent results can be found in [LL50, Mik64b, Was68, Rek77, HHNL88, You69] and others.

The application of computers and numerical methods to the solution of engineering problems gave a powerful stimulus to the development of variational approaches. Classical methods were revised and new methods were created for extremal problems under unilateral constraints. Such problems arise in mathematical programming, optimization and control, see, e.g., [Roc70, Lio68, Ban83a, FM68, Fle81, NST06, HN96] and many others.

Important theoretical results were, simultaneously, obtained for these new problems in physical sciences, mechanics, biology, and other areas. The contributions of French and Italian mathematicians (J.-L. Lions, G. Duvaut, G. Stampacchia, J. Céa, D. Kinderlehrer, R. Glowinski) must be noted in this connection, see [DL72, KS80, Céa64, Glo84, GLT81]. Considerable successes were attained in the work of P. D. Panagiotopoulos [AP92, Pan85] devoted to the contact problems in the mechanics of solids.

Note that the contact problems in the classical theory of elasticity first were solved analytically, with the potential theory method [Her95], complex variables methods [Mus53] and others. For a survey of these methods and the analytical solutions, see [Gla80].

One can conclude from the analysis of the contents of these (and other) books and articles that few publications exist on problems such as variational formulations of contact with friction, algorithms for their solution and convergence theorems, contact with adhesion, and numerical solutions to technical problems. In addition, the modern state of the theory and applications of solutions to unilateral problems permits the inclusion of these topics in academic courses. Therefore, it is needed a textbook which contains all the necessary mathematical tools. These arguments motivated the creation of this book.

The book contains eight chapters. Bearing in mind the possible use of the book as a textbook, the authors proceed from simple examples to general theory. We give the notations and mathematical tools in Chapter 1. In Chapter 2, the equilibrium of linear systems is considered for finite-dimensional systems, continuous systems, and to linear elastic bodies.

Chapter 3 is devoted to nonlinear smooth systems without unilateral constraints. It considers the differentiation of functionals and operators, extremal conditions, existence and uniqueness theorems for minimization problems, and operator potentiality conditions. Two examples from the mechanics of solids are considered: boundary value problems (BVP) for the Hencky–Ilyushin theory of plasticity without discharge and BVP for nonlinear elastic bodies with finite displacements and strains.

Problems with unilateral constraints are investigated in Chapter 4. Contact interaction of deformed bodies with smooth contact surfaces and with finite friction are considered. It is demonstrated that the local frictionless contact problem is equivalent to the minimization of a functional via a variational inequality. This statement is generalized to a system of deformed bodies with new results on the influence of the different forms of the impenetrability condition. Generalizations on nonlinear governing equations are proposed, including processes with finite displacements and strains.

Laws of Coulomb-type governing friction include the relative sliding velocity of particles on contact surfaces. Admissible displacement and velocity fields are constructed with the Ostrogradski method. It is found that the corresponding variational formulation is a quasi-variational inequality. An iterative method for the solution of this inequality is proposed. An *a priori* solution estimate is given as well as the foundation of the transition from velocities to displacements in the governing law.

Chapter 5 is devoted to transformations of variational problems with unilateral constraints. Following the presentation method, the simplest problems without unilateral constraints are considered first. For such problems (BVP for an ordinary differential equation and BVP for the Poisson equation) the classical Friedrichs transformation is adequate and permits solutions to the dual problems as well as to the saddle-point problems. This method permits us to find the equilibrium and the mixed and hybrid variational principles [BF91].

It is well known that the generalization of the Friedrichs transformation to problems with unilateral constraints was made by Young, Fenchel, and Moreau. This transformation was applied to contact problems and enables one to obtain a set of variational problems including some new variational principles.

In Chapter 6, the variational principles and methods for nonstationary problems are considered, including variational principles for dissipative systems and the Gurtin method for the dynamic elasticity problem and for viscoelastic BVP. Here, a description of contact interaction with adhesion is given.

Chapter 7 is devoted to solutions method, algorithms, and their numerical implementation. The finite element and BEMs are presented. The solution of some model problems as well as important technical ones are given for 2D and 3D formulations.

In conclusion (Chapter 8), we outline some modern research directions in the variational theory of unilateral problems, including relations of this theory with identification and optimization problems, history and recent results obtained in the theory of contact problems with friction, wear phenomena, large displacements and deformations, adhesion which plays an important role in micromechanics and nanomechanics of deformed solids. A survey is given on the numerical solution of unilateral problems as well, using the finite element and BEMs.

It must be emphasized that the variational approach has been developed rapidly in the last few years and has been successfully used in medicine, biology, economics, heat conduction modeling, tomography, and many others branches of science and technology.

The starting point of our work on unilateral problems was a collaboration with Professors R. Glowinski and P. A. Raviart. We are happy to acknowledge their contributions.

We thank all our colleagues for fruitful discussions on many aspects of unilateral problems, in particular, Professors J. Haslinger, B. E. Pobedrya, N. V. Banichuk and the Full Member of the Russian Academy of Sciences I. G. Goryacheva.

We appreciate the considerable contributions of the editor of our book, Professor G. M. L. Gladwell, for his numerous remarks and suggestions on the content of the book and exposition of the results. We express our sincere thanks to M.-L. Rantalainen for her excellent technical assistance and J. Räbinä for graphical illustration.

Part of the research was supported by grants RFFI, No. 05-01-000591, 05-08-08085, by Academy of Finland, and by Tekes' (Finnish Funding Agency for Technology and Innovation) MASI programme.

Jyväskylä, *Alexander Kravchuk*
January 2007 *Pekka Neittaanmäki*

Contents

1

Notations and Basics

We use in this chapter essentially the monograph [LVG02] and [Ped89, Ger73, LM73, Mik64b, Neč67, Sob50, Sok64, Yos65] as well. Notations for the vectors and tensors in an arbitrary coordinate system are the same as in [GA60].

1.1 Notations and conventions

1.1.1 Space of independent variables

$\mathbb{R}^n$ is an Euclidean space of n dimensions. $(k_1, k_2, .., k_n) = \{k_i\}_{i=1}^n$ is a Cartesian basis in $\mathbb{R}^n$. $(G_1, G_2, \ldots, G_n) = \{G_i\}_{i=1}^n$ is an arbitrary basis in $\mathbb{R}^n$. An element in $\mathbb{R}^n$ is notated by x or a.

1.1.2 Tensors and vectors

Tensor of the second order is $\hat{t}, \hat{s}, \ldots$, tensor of the third order ${}^3\hat{a}$, and tensor of the nth order ${}^n\hat{a}$. For vectors we have no special notations.

1.1.3 Summation notation

$V_i \delta_{ij} V_j$ means $\sum_{i,j} V_i \delta_{ij} V_j$. For example, decomposition of a vector in the Cartesian basis $\{k_i\}_{i=1}^n$ is

$$a = a_i k_i \tag{1.1}$$

and decomposition of a vector in an arbitrary basis $\{G_i\}_{i=1}^n$

$$a = a^i G_i = a_i G^i, \tag{1.2}$$

where a^i is the *contravariant* component of the vector a and a_i is the *covariant* component of the vector a. A contravariant component is denoted with an upper index and a covariant component is denoted with a lower index.

(Note that, in general, the components a_i in (1.1) are not equal to the components a_i in (1.2). See also Remark 1.1 and the definitions below.)

By definition:

$$G^i = G^{ij} G_j, \quad G_{ij} = G_i \cdot G_j,$$

where $G_i \cdot G_j$, is the inner (scalar) product (see the formula (1.3)), (G^{ij}) is the matrix inverse to the matrix G_{ij}, i.e., $G^{ij} G_{jk} = \delta^i_k$ where $\delta^i_k = 1$ for $i = k$, $\delta^i_k = 0$ for $i \neq k$ is the *Kronecker symbol*. In a Cartesian basis we will use the notation δ_{ik} for the Kronecker symbol because in such a basis a contravariant and covariant components are equal, and the position of the indices is insignificant.

Remark 1.1. The formulae (1.1) and (1.2) for the decomposition of a vector a are similar to the formula (1.3) for the inner product, but these sums have different senses. The first two are the decompositions of a vector, the second is the scalar product. There is no ambiguity in these formulae because the meaning is clear from the context.

$V_{\underline{ii}}$ means that there is no summation.

1.1.4 Algebraic operation

A *scalar product* of the vectors is $a \cdot b$. In the Cartesian basis $\{k_i\}_{i=1}^n$

$$a \cdot b = a_i b_i. \tag{1.3}$$

In an arbitrary basis $\{G_i\}_{i=1}^n$

$$a \cdot b = a^i b_i = G_{ij} a^i b^j = G^{ij} a_i b_j. \tag{1.4}$$

A *vector product* of the vectors $a \times b$ in the Cartesian basis $\{k_i\}_{i=1}^3$ (we do not use the vector product for an arbitrary n) is

$$a \times b = (\varepsilon_{ijl}) a_j b_l k_i, \tag{1.5}$$

where ε_{ijl} is the *Levi-Civita* symbol defined as follows:

$$\varepsilon_{ijl} = \begin{cases} -1, & \text{if } i, j, l \text{ is even,} \\ +1, & \text{if } i, j, l \text{ is odd,} \\ 0, & \text{if at least two for } i, j, l \text{ coincide.} \end{cases}$$

A sequence i, j, l is said to be even (odd) if it can be obtained from $1, 2, 3$ by an even (odd) number of permutation.

The formula for the vector product in an arbitrary basis is obtained by the transformation from a Cartesian basis to an arbitrary basis $\{G_i\}_{i=1}^3$:

$$a \times b = (E_{ijl}) a^j b^l G_i \tag{1.6}$$

where

$$E_{ijl} = G_i \cdot (G_j \times G_l),$$

and the product $G_i \cdot (G_j \times G_l)$ is calculated by (1.5).

A *tensor (dyadic) product* of the vectors is $a \otimes b$. In the Cartesian basis $\{k_i\}_{i=1}^n$

$$a \otimes b = a_i b_j k_i \otimes k_j, \tag{1.7}$$

where $k_i \otimes k_j$ is the matrix with the entry in row i and column j equal to 1, and all the other entries are zero.

In an arbitrary basis $\{G_i\}_{i=1}^n$:

$$a \otimes b = a_i b_j G^i \otimes G^j = a^i b^j G_i \otimes G_j = a^i b_j G_i \otimes G^j = a_i b^j G^i \otimes G_j \tag{1.8}$$

and the tensor products of the basic vectors G_i, G^j are calculated by (1.7).

Using the operations, (1.7) and (1.8), we can define the decomposition of the tensor $\hat{t}$:

$$\hat{t} = t_{ij} k_i \otimes k_j = t_{ij} G^i \otimes G^j = t^{ij} G_i \otimes G_j = t_{i\cdot}^{\cdot j} G^i \otimes G_j = t_{\cdot j}^{i \cdot} G_i \otimes G^j. \tag{1.9}$$

If $t_{ij} = t_{ji}$, then $t^{ij} = t^{ji}$, $t_{i\cdot}^{\cdot j} = t_{j\cdot}^{\cdot i} = t_{\cdot j}^{i\cdot} = t_{\cdot i}^{j\cdot} \equiv t_i^j$, and the tensor $\hat{t}$ is *symmetric*. So, for the symmetric tensor we do not use "$\cdot$" as an index. We emphasize that, in general, the values of the components t_{ij}, t^{ji}, $t_{i\cdot}^{\cdot j}$, $t_{\cdot j}^{i\cdot}$ are different.

Using the decomposition (1.9), we define the scalar and vector product of a tensor and vector as

$$\begin{aligned} a \cdot \hat{t} &= t_{ij}(a \cdot k_i) k_j = \ldots, \\ \hat{t} \cdot a &= t_{ij} k_i (a \cdot k_j) = \ldots, \\ a \times \hat{t} &= t_{ij}(a \times k_i) \otimes k_j, \ldots, \\ \hat{t} \times a &= t_{ij} k_i \otimes (k_j \times a), \ldots, \end{aligned} \tag{1.10}$$

Dots means that the results of an operation can be calculated in an arbitrary basis $\{G_i\}_{i=1}^n$. The decomposition (1.9) and operations (1.10) are generalized on a tensor of an arbitrary order (see, e.g., [McC57]).

Tensor convolution for tensors of the second order in the Cartesian basis $\{k_i\}_{i=1}^n$ is

$$\hat{t} \cdot \cdot \hat{s} = t_{ij} s_{ji}. \tag{1.11}$$

In an arbitrary basis $\{G_i\}_{i=1}^n$

$$\hat{t} \cdot \cdot \hat{s} = t^{ij} s_{ji} = t_{ij} s^{ji} = t^{ij} s^{pq} G_{jp} G_{iq} = t_{ij} s_{pq} G^{jp} G^{iq} = t_i^j G^i G_j = t_j^i G_i G^j. \tag{1.12}$$

We will use the convolution ${}^4\hat{a} \cdot \cdot \hat{\varepsilon}$. In the Cartesian basis $\{k_i\}_{i=1}^n$

$${}^4\hat{a} \cdot \cdot \hat{\varepsilon} = a_{ijpq} \varepsilon_{qp} k_i \otimes k_j. \tag{1.13}$$

(We do not use this operation in an arbitrary basis.)

1.1.5 Differential operator notation

The *Hamilton operator* ∇ is defined as the vector-operator. In the Cartesian basis $\{k_i\}_{i=1}^n$

$$\nabla = \frac{\partial}{\partial a_i} k_i. \tag{1.14}$$

In the Cartesian basis we define the following operators:

- *Divergence operator:*
$$\operatorname{div} \hat{t} = \nabla \cdot \hat{t}, \tag{1.15}$$
- *Gradient operator:*
$$\operatorname{grad} \hat{t} = \nabla \otimes \hat{t}, \tag{1.16}$$
- *Rotor (curl) operator:*
$$\operatorname{curl} \hat{t} = \nabla \times \hat{t}. \tag{1.17}$$

Note that the gradient operator is defined for the scalar, vector, and a tensor of an arbitrary order. Divergence and curl operators are defined for a vector and a tensor of arbitrary order.

In problems of deformed body mechanics, curvilinear coordinate systems are used. Such systems are generated by displacements of the particles of the body from their initial positions a to current positions x. In the Cartesian basis (in such a basis the position of an index is insignificant, and we now use upper indices):

$$x^i = x^i(a^1, a^2, a^3). \tag{1.18}$$

This formula can be interpreted as the introduction of a curvilinear coordinate system in the domain Ω: the coordinate line $a^2 = \text{const}$, $a^3 = \text{const}$, $a^1 \neq \text{const}$, being in the domain Ω a straight line which is parallel to the vector k_1. As a result of the transformation (1.18), this line is transformed to some curved line in the domain Ω. The collection of three such coordinate lines in the domain Ω, existing at every point due to the definition, is a system of curvilinear coordinates in this domain. For this given curvilinear coordinate system at every point of the domain Ω we define three vectors G_i, being tangent to the coordinate lines

$$G_1 = \frac{\partial x}{\partial a^1}, \quad G_2 = \frac{\partial x}{\partial a^2}, \quad G_3 = \frac{\partial x}{\partial a^3}. \tag{1.19}$$

In such a system, we define the Hamilton operator

$$\nabla = G^p \nabla_p, \tag{1.20}$$

where G^j are calculated by (1.19). A covariant component ∇_p of the vector-operator ∇ allows us to take into account the variation of the derivative of a vector field $Q = Q(x) = Q(x(a))$ due to variations of the basis $\{G_i\}_{i=1}^3$:

$$\nabla_i Q_j = \frac{\partial Q_j}{\partial a^i} - Q_k \Gamma_{ij}^k. \tag{1.21}$$

This formula defines the so-called *covariant derivative of the covariant components* of the vector Q. The quantities Γ_{ij}^k are called the *Christoffel symbols of the second kind.*

With the definitions (1.20) and (1.21) we can define all the differential operators in the basis $\{G_i\}_{i=1}^n$ for a tensor of second-order $\hat{t}$ (with the generalizations formulated above for a Cartesian coordinate system):

- *Divergence operator:*
$$\operatorname{div} \hat{t} = \nabla \cdot \hat{t}, \tag{1.22}$$
- *Gradient operator:*
$$\operatorname{grad} \hat{t} = \nabla \otimes \hat{t}, \tag{1.23}$$
- *Rotation (curl) operator:*
$$\operatorname{curl} \hat{t} = \nabla \times \hat{t}. \tag{1.24}$$

We formulate the *generalized Gauss–Ostrogradski theorem* (used in the transformations of the boundary value problems (BVPs) to variational ones).

Theorem 1.2. *It holds*
$$\int_\Omega \nabla \circ \hat{T}\, d\Omega = \int_\Sigma \nu \circ \hat{T}\, d\Sigma, \tag{1.25}$$
where "$\circ$" is one of the operations in (1.22)–(1.24), ν is the unique outward drawn normal vector to the surface Σ.

For hypotheses on the domain Ω and its boundary Σ see Definition 1.65. Applications of Theorem 1.2 will be given in Chapter 3.

1.2 Functional spaces

We begin with the consideration of the set of all the vectors $x = (x_1, x_2, \ldots, x_n)$, where x_i, $i = 1, 2, \ldots, n$ are real numbers. This set is called $\mathbb{R}^n$. Every vector $x = (x_1, x_2, \ldots, x_n)$ can be interpreted as a point in $\mathbb{R}^n$. The *Euclidean distance* $\rho_E(x, y)$ between two points with coordinates $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is defined as the length of the vector $(x - y)$, i.e.,
$$\rho_E(x, y) = |x - y| = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

1.2.1 Open, closed, and compact sets in $\mathbb{R}^n$

Let $\{x^{(k)}\}_{k=1}^\infty$ be a sequence of the vectors (points) in $\mathbb{R}^n$.

Definition 1.3. *The sequence* $\{x^{(k)}\}_{k=1}^\infty$ converges *to* $x \in \mathbb{R}^n$ *if, for given* $\varepsilon > 0$*, we can find an integer* N_ε *such that for all* $k > N_\varepsilon$ *we have* $\rho_E(x^{(k)}, x) < \varepsilon$*.*

Definition 1.4. *The* open ball *with center $x^{(0)}$ and radius a in $\mathbb{R}^n$ is the set such that*

$$x \in \mathbb{R}^n, \quad \rho_E(x^{(0)}, x) < a.$$

Definition 1.5. *A set $S \subset \mathbb{R}^n$ is* open *if every point of S is the center of an open ball lying entirely in S.*

Definition 1.6. *A set $S \subset \mathbb{R}^n$ is said to be* closed *if every convergent sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset S$ converges to a point $x \in S$. The point $x \in S$ is the* limit point *of the sequence $\{x^{(k)}\}$.*

Definition 1.7. *A set $S \subset \mathbb{R}^n$ is said to be* compact *if every sequence $\{x^{(k)}\}_{k=1}^{\infty} \subset S$ contains a subsequence converging to a point $x \in S$.*

Theorem 1.8. *A set $S \subset \mathbb{R}^n$ is compact iff (we use the term* iff *to denote "if and only if") it is closed and bounded (contained in a ball with radius $a < \infty$).*

Definition 1.9. *The* closure $\bar{S}$ *of a set S is the set obtained by adding to S the limit points of all the convergent sequences $\{x^{(k)}\}_{k=1}^{\infty} \subset S$.*

We use the symbol Ω, and the term *domain* to denote a nonempty open set in $\mathbb{R}^n$. A rule which assigns a unique real number to every $x \in \Omega$ is said to define a *real function* $f(x)$ on Ω.

Definition 1.10. *The* support *of $f(x)$ in Ω, written* $\operatorname{supp} f$, *is defined as*

$$\operatorname{supp} f = \overline{\{x \in \Omega \mid f(x) \neq 0\}},$$

where the overbar means closure in $\mathbb{R}^n$. The function $f(x)$ is said to have compact support *if* $\operatorname{supp} f$ *is bounded, i.e., contained in some ball in $\mathbb{R}^n$. It is said to have compact support in Ω if* $\operatorname{supp} f \subset \Omega$.

1.2.2 Metric spaces

We use real functional spaces only. Let X be a set of an arbitrary elements (points) x.

Definition 1.11. *A metric space is a pair (X, ρ) consisting of a set X together with a metric ρ, a real valued function $\rho(x, y)$ defined for any two points $x, y \in X$, which satisfies the following hypotheses:*

(H1) $\rho(x, y) \geq 0$;
(H2) $\rho(x, y) = 0$ *iff* $x = y$;
(H3) $\rho(x, y) = \rho(y, x)$;
(H4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

A real valued function $\rho(x, y)$ is called a metric for X.

Let $X = \mathbb{R}^n$, with the metric

$$\rho(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

and let $\{x^{(k)}\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, an element $x^{(k)}$ is a vector with the components $x_i^{(k)}$, $i = 1, \ldots n$.

Definition 1.12. *An infinite sequence $\{x^{(k)} \in X\}$ is said to converge to x under the metric ρ if, for given $\varepsilon > 0$, there exists an integer N_ε such that for all $k > N_\varepsilon$ we have $\rho(x, x^{(k)}) < \varepsilon$. We write $x^{(k)} \to x$, and have*

$$\lim_{k\to\infty} \rho(x, x^{(k)}) = 0$$

or

$$\lim_{k\to\infty} (x^{(k)}) = x.$$

We give now some examples:

1. l^p is the set of all sequences in $\mathbb{R}^\infty$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. The metric is
$$\rho(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}.$$
2. l^∞ is the metric space of all bounded sequences in $\mathbb{R}^\infty$. The metric is
$$\rho(x,y) = \sup_i \{|x_i - y_i|\}.$$

1.2.3 Sets in a metric space

Definition 1.13. *In a metric space X the set of points*

$$B = \{x \in X \mid \rho(x^{(0)}, x) < \varepsilon\}$$

is called the open ball *of radius ε about $x^{(0)}$. We also call it an ε-*neighborhood *of $x^{(0)}$ and denote it by $O_\varepsilon(x^{(0)})$. $x^{(0)}$ is called the center of the ε-neighborhood. A* neighborhood *of $x^{(0)}$ is any subset Ξ of X which contains an ε-neighborhood B of $x^{(0)}$. Conversely, we call $x^{(0)}$* an interior point *of a set $\Xi \subset X$ if Ξ is a neighborhood of $x^{(0)}$.*

Definition 1.14. *A set S in a metric space X is said to be* open *if every point $x \in X$ is the center of an ε-*neighborhood *of radius $\varepsilon(x)$ contained in S. Thus, every point of an open set S is an interior point.*

Definition 1.15. *A point $x \in X$ is called a* contact point *of a set $S \subset X$ if every neighborhood of x contains at least one point of S, maybe just x. The set of all contact points is the* closure *of S and is denoted by $\bar{S}$, see Definition 1.9. Clearly $S \subset \bar{S}$, since every point of S is a contact point.*

Definition 1.16. *A point* $x \in X$ *is called a* limit point *or* accumulation point *of a set* S *if every neighborhood of* x *contains an infinity of points of* S*. The limit point may or may not belong to* S*.*

Definition 1.17. *Let* X *be a metric space. A set* S *is said to be* closed *in* X *if* $S = \bar{S}$*, i.e., if it contains all its contact points, and in particular, all its limits points.*

Definition 1.18. *Suppose* S *and* T *are two sets such that* $S \subseteq T \subseteq X$*. The set* S *is said to be dense in* T *if* $T \subset \bar{S}$*. By definition* S *is dense in* $\bar{S}$*.*

Define now some functional metric spaces.

Definition 1.19. $C(\Omega)$ *is the set of continuous functions on* Ω*.*

Definition 1.20. $C_B(\Omega)$ *is the subset of* $C(\Omega)$ *consisting of functions which are bounded on* Ω*).*

If in $C_B(\Omega)$

$$\rho(f, g) = \sup_{x \in \Omega} |f(x) - g(x)|,$$

then we have a metric space.

Definition 1.21. $C_c(\Omega)$ *is the subset of* $C(\Omega)$ *consisting of functions of compact support in* Ω*.*

Definition 1.22. *A function* $f(x)$ *is said to be* uniformly *continuous on* Ω *if, for given* $\varepsilon > 0$*, we can find* $\delta > 0$ *such that, if* $x, y \in \Omega$ *and* $\rho_E(x, y) < \delta$*, then* $|f(x) - f(y)| < \varepsilon$ *for all* $x, y \in \Omega$*.*

Theorem 1.23. *If a function* $f(x)$ *is continuous on* $\bar{\Omega}$*, then it is uniformly continuous.*

The set of those functions is denoted by $C(\bar{\Omega})$. In this book, we deal with the derivatives of functions and we must have some way of measuring the distance between the derivative of two functions.

Introduce the abbreviation

$$D^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \ldots \partial x_n^{k_n}}, \quad |k| \equiv k = k_1 + k_2 + \ldots + k_n. \tag{1.26}$$

Definition 1.24. *Let* m *be a nonnegative integer.* $C^m(\Omega)$ *is the set of functions* $f(x)$*,* $x \in \Omega$ *which have continuous derivatives* $D^k f$ *for all* $|k| \le m$*.*

We can use as the metric

$$\rho(f, g) = \sum_{|k| \le m} \sup_{x \in \Omega} |D^k f - D^k g| \tag{1.27}$$

or

$$\rho(f,g) = \max_{|k|\le m} \sup_{x\in\Omega} |D^k f - D^k g|. \tag{1.28}$$

To be sure that the quantities (1.27) and (1.28) are finite, we must use special subsets of $C^m(\Omega)$.

As above, we can introduce metric spaces $C_B^m(\Omega)$, $C_c^m(\Omega)$, and $C^m(\bar{\Omega})$. For example, for $C_B^m(\Omega)$ we take the subset of $C^m(\Omega)$ for which $|D^k f|$, $|k| \le m$ are bounded on Ω.

Definition 1.25. *We define $C^\infty(\Omega)$ to be a set of functions having continuous derivatives of all orders on Ω, i.e., $C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$ and C_c^∞ is the subset of $C^\infty(\Omega)$ of functions having compact support.*

Definition 1.26. *A sequence $x^{(k)}$ in a metric space X is said to be a* Cauchy sequence *if, for given $\varepsilon > 0$, there exists N_ε depending on ε such that, if the integers $p, q \ge N_\varepsilon$, then $\rho(x^{(p)}, x^{(q)}) < \varepsilon$.*

Definition 1.27. *A metric space X is said to be* complete *if any Cauchy sequence in X has a limit in X. Otherwise, it is said to be* incomplete.

Definition 1.28. *A set S is said to be dense in a metric space X if any ε-neighborhood of $x \in X$ contains a point $s \in S$.*

Definition 1.29. *A correspondence between two metric spaces (X_1, ρ_1) and (X_2, ρ_2) is said to be* one-to-one *if there is a rule which assigns a unique element $x_2 \in X_2$ to each element $x_1 \in X_1$ and vice versa.The correspondence is said to be* isometric *if*

$$\rho_1(x_1, y_1) = \rho_2(x_2, y_2).$$

Theorem 1.30. *For a metric space X, there is an isometric one-to-one correspondence between X and a set $\tilde{X}$ which is dense in a complete metric space $\hat{X}$, called the* completion *of X.*

Definition 1.31. *Let X and Y be metric spaces. A correspondence $A(x) = y$, $x \in X$, $y \in Y$ is called an* operator from X into Y *if to each $x \in X$ there corresponds no more than one $y \in Y$. The set of all those $x \in X$ for which there exists a corresponding $y \in Y$ is called the* domain *of A and denoted by $D(A)$. The set of all y arising from $x \in X$ is called the* range *of A and denoted by $R(A)$. Thus*

$$R(A) = \{y \in Y \mid y = A(x),\ x \in X\}.$$

We say that A is an operator on $D(A)$ into Y, or on $D(A)$ onto $R(A)$. We also say that $R(A)$ is the image *of $D(A)$ under A.*

Definition 1.32. *A* functional *is a particular case of an operator, in which $R(A) \subset \mathbb{R}$.*

Definition 1.33. *Let A be an operator from X into Y. The operator A is said to be* continuous *at $x_0 \in X$ if, for given $\varepsilon > 0$, there is $\delta > 0$, depending on ε, such that if $\rho_X(x, x_0) < \delta$ then $\rho_Y(A(x), A(x_0)) < \varepsilon$.*

Definition 1.34. *An operator A acting on a metric space is called a* contraction operator *or a* contracting mapping *in X, provided there exists a real number q with $0 \le q < 1$ such that*

$$\rho(A(x), A(y)) \le q\rho(x, y).$$

Consider an operator equation

$$A(x) = x \tag{1.29}$$

arising in BVPs of mechanics considered in this book.

Theorem 1.35 (Banach fixed point theorem). *Let A be a contraction operator in a complete metric space X. Then*

(i) A has only one fixed point $x_* \in X$.
(ii) For any initial approximation $x^{(0)} \in X$, the sequence of successive approximations

$$x^{(k+1)} = A(x^{(k)}), \quad k = 0, 1, \dots,$$

converge to x_*, the solution to (1.29). The rate of convergence is estimated by

$$\rho(x^{(k)}, x_*) \le \frac{q^k}{1-q}\rho(x^{(0)}, x^{(1)}). \tag{1.30}$$

This theorem is used to the demonstration of existence theorems and, of course, for the justification of different successive approximation methods.

1.2.4 Normed linear spaces

We suppose the reader is familiar with the definition of a linear set (space).

Definition 1.36. *A quantity $\|x\|$ is called a* norm *in a linear space X if it is a real valued function defined for every $x \in X$ which satisfies the following* norm axioms*:*

(H1) $\|x\| \ge 0$, *and* $\|x\| = 0$ *iff* $x = 0$.
(H2) $\|\lambda x\| = |\lambda|\|x\|$ *when* $\lambda \in \mathbb{R}$.
(H3) $\|x + y\| \le \|x\| + \|y\|$.

Definition 1.37. *A linear space X is called a* normed linear space *if, for every $x \in X$, a norm $\|x\|$ satisfying (H1)–(H3) is defined.*

In a normed linear space we can define a metric

$$\rho(x, y) = \|x - y\|. \tag{1.31}$$

Then the normed linear space is a metric space, and all the definitions introduced for a metric space can be applied for a normed space, i.e., the definition of a complete space, closed space, etc.

Definition 1.38. *A space Y is said to be a* linear subspace *of a linear space X if Y is linear space and Y is a subset of X.*

Definition 1.39. *Let X be a normed linear space, and suppose $Y \subset X$. The set Y is said to be a* normed linear subspace *of X if Y is a linear space, equipped by the norm on Y. This norm on Y is said to be* induced *by the norm on X.*

Definition 1.40. *Let X be a normed linear space, and $Y \subset X$. Y is called a* closed subspace *of X if it is closed as a set with the norm induced by X, and it is a subspace of X.*

Definition 1.41. *Let X be a linear space. The elements $x_1, x_2, \ldots, x_m$ are said to be* linearly independent *if the equation*

$$\sum_{i=1}^{m} c_i x_i = 0$$

implies $c_1 = c_2 = \ldots = 0$. Otherwise $x_1, x_2, \ldots, x_m$ are said to be linearly dependent. *In this case there is at least one of the x_i which can be expressed as a linear combination of the others.*

Definition 1.42. *The linear space is said to be finite-dimensional if there is a positive integer n such that X contains n linearly independent elements, but any set of $n+1$ elements is linearly dependent. We write* $\dim(X) = n$. *If X is not finite-dimensional we shall say that it is* infinite-dimensional.

Definition 1.43. *A complete normed linear space is called a* Banach space.

Theorem 1.44. *1. The space $C(\bar{\Omega})$ with the norm*

$$\|f\|_{0,\infty,\Omega} = \sup_{x \in \Omega} \|f(x)\| \tag{1.32}$$

is a Banach space.

2. The space $C(\bar{\Omega})$ with the norm

$$\|f\|_{m,\infty,\Omega} = \max_{|k| \le m} \sup_{x \in \Omega} |D^k f(x)| \tag{1.33}$$

is a Banach space.

Definition 1.45. *The operator A is a* linear operator *from X into Y if its domain is a linear subspace of X and, for every $x_1, x_2 \in D(A)$ and every $\alpha, \beta \in \mathbb{R}$,*

$$A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2).$$

Suppose that X and Y are normed linear spaces and A is a linear operator from X into Y. The operator A has a domain $D(A)$ which is a subspace of X. A range $R(A)$ is a subspace of Y, and a null space $N(A)$ consists of $x \in X$ such that $Ax = 0$, which also is a subspace of X.

Theorem 1.46. *The operator A is continuous on $D(A)$ iff there is a constant c such that, for all $x \in D(A)$, we have*

$$\|Ax\|_Y \leq c\|x\|_X. \tag{1.34}$$

Definition 1.47. *The infimum of such constants c is called the* norm *of A and it is denoted by $\|A\|$.*

From Definition 1.47 it follows that

$$\|Ax\|_Y \leq \|A\|\|x\|_X. \tag{1.35}$$

Formulate some useful inequalities (see, e.g., [LVG02]):

1. Let X be a normed linear space and $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of elements in X. Then, there is a number $c > 0$ such that for every set of scalars $(c_1, c_2, \ldots, c_n)$ we have the inequality:

$$\|c_1 x_1 + c_2 x_2 + \ldots + c_n x_n\| \geq c(|c_1| + |c_2| + \ldots + |c_n|). \tag{1.36}$$

2. Let $1 < p < \infty$, $1/p + 1/q = 1$, $a > 0$, $b > 0$. Then

$$ab \leq a^p/p + b^q/q$$

   with equality iff $a^p = b^q$.
3. Let $1 < p < \infty$, $1/p + 1/q = 1$, $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b_2, \ldots, b_n \geq 0$. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{1.37}$$

   This inequality is the *Hölder* inequality. In a normed spaces the Hölder inequality is as follows [Ped89]:

$$\int_\Omega |fd|\, d\Omega \leq \left(\int_\Omega |f|^p\, d\Omega\right)^{1/p} \left(\int_\Omega |g|^q\, d\Omega\right)^{1/q}. \tag{1.38}$$

4. Let $1 \leq p < \infty$, $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b_2, \ldots, b_n \geq 0$. Then

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p} \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}. \tag{1.39}$$

This inequality is the *Minkowski* inequality. The Minkowski inequality for a functions in a normed spaces is as follows:

$$\left(\int_\Omega |f(x)+g(x)|^p\, d\Omega\right)^{1/p} \le \left(\int_\Omega |f(x)|^p\, d\Omega\right)^{1/p} + \left(\int_\Omega |g(x)|^p\, d\Omega\right)^{1/p}. \tag{1.40}$$

1.2.5 Inner product spaces, Hilbert spaces, and Lebesgue spaces

Definition 1.48. *Let X be a linear space. The function (x, y), uniquely defined for every pair $x, y \in X$, is called an* inner product *on X if it satisfies the following axioms:*

(H1) $(x,x) \ge 0$ *and* $(x,x) = 0$ *iff* $x = 0$.
(H2) $(x,y) = (y,x)$.
(H3) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$, *where* $\lambda, \mu \in \mathbb{R}$. *A linear space X with an inner product is called an* inner product space.

With the norm

$$\|x\| = (x,x)^{1/2} \tag{1.41}$$

a linear inner product space is a linear normed space.

For any $x, y \in X$ the following inequality holds:

$$|(x,y)| \le \|x\|\|y\| \tag{1.42}$$

where, for $x \ne 0$, $y \ne 0$, equality occurs iff $x = \lambda y$. Inequality (1.42) is the *Cauchy–Buniakowski* or the *Schwartz inequality.*

Definition 1.49. *We say that x and y are* orthogonal *if* $(x,y) = 0$.

Definition 1.50. *Let X be a linear inner product space. A* closed subspace *of X is a set S which is a subspace of X and which, as a set, is closed under the metric corresponding to the inner product induced on S.*

Definition 1.51. *A complete inner product space is called a* Hilbert space *and is denoted by H.*

Definition 1.52. *Let Ω be a domain in $\mathbb{R}^n$, and let $C_I(\Omega)$ be a subspace of the function of $C(\Omega)$ satisfying the inequality*

$$\int_\Omega |f(x)|^p\, d\Omega < \infty \tag{1.43}$$

where $1 \le p < \infty$.

The Lebesgue space $L^p(\Omega)$ *is the completion of this subspace in the norm*

$$\|f\|_p \equiv \|f\|_{0,p,\Omega} \equiv \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, d\Omega\right)^{1/p}. \tag{1.44}$$

By definition (see above), the sequence $\{f_k(x)\} \subset C_I(\Omega)$ is a Cauchy sequence in $\|\cdot\|_p$ if

$$\|f_k - f_l\|_p = \left(\int_\Omega |f_k(x) - f_l(x)|^p \, d\Omega\right)^{1/p} \longrightarrow 0 \quad \text{as } k, l \to \infty, \tag{1.45}$$

and two sequences $\{f_k(x)\}$ and $\{g_k(x)\}$ are equivalent if

$$\|f_k - g_k\|_p = \left(\int_\Omega |f_k(x) - g_k(x)|^p \, d\Omega\right)^{1/p} \longrightarrow 0 \quad \text{as } k \to \infty. \tag{1.46}$$

Note that the elements of $L^p(\Omega)$ can be considered as equivalent classes of Cauchy sequences of functions [LVG02]. These classes will be denoted by $F(x)$.

Definition 1.53. *The sequence $\{f_k(x)\} \subset C_I(\Omega)$ (see Definition 1.52) is said to be a* null sequence *if $\|f_k\|_p \to 0$ as $k \to \infty$.*

Definition 1.54. *The function $f(x)$ is said to be* zero almost everywhere *(we write $f(x) = 0$ a.e.) if there is a null sequence $\{f_k(x)\} \subset C_I(\Omega)$ such that*

$$f(x) = \lim_{k\to\infty} f_k(x), \quad x \in \Omega. \tag{1.47}$$

For such functions

$$\|f(x)\|_p = \lim_{k\to\infty} \|f_k(x)\|_p = 0. \tag{1.48}$$

Theorem 1.55 (Embedding theorem). *Suppose that Ω is a bounded domain, and p, q satisfy $1 \le p \le q < \infty$. If $F(x) \in L^q(\Omega)$, then $F(x) \in L^p(\Omega)$ and*

$$\|F\|_{0,p,\Omega} \le (\text{mes}\,\Omega)^{1/p-1/q}\|F\|_{0,q,\Omega}.^1 \tag{1.49}$$

In this case we say that $L^q(\Omega)$ is embedded *in $L^p(\Omega)$ and we write this*

$$L^q(\Omega) \longrightarrow L^p(\Omega), \quad 1 \le p \le q < \infty.$$

The embedding defines an *embedding operator* I which is a linear and continuous operator. Embedding can be generalized as follows.

Definition 1.56. *We say that the normed space X is embedded in the normed space Y, and write this $X \to Y$, if X is a subspace of Y and the operator I from X to Y defined by $Ix = x$ for all $x \in X$ is continuous.*

[1] mes Ω in mechanics is equal to the volume occupied by Ω in the space $\mathbb{R}^n$, $n = 1, 2, 3$. For the definition of mes Ω in an abstract space see, e.g., [Ped89].

1.2.6 Generalized derivatives, Sobolev spaces

Let Ω be a domain, a nonempty open set, in $\mathbb{R}^n$. Recall the definition:

$$D^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}, \quad |k| \equiv k = k_1 + k_2 + \dots + k_n.$$

Definition 1.57. *The quantity $D^k f$ is called a generalized derivative if the relation*

$$\int_\Omega D^k f \phi \, d\Omega = (-1)^k \int_\Omega f D^k \phi \, d\Omega \tag{1.50}$$

holds for any $\phi \in \mathcal{D}(\Omega)$ where $\mathcal{D}(\Omega)$ is the space of functions $C^\infty(\Omega)$ (being the set of functions having continuous derivatives of all orders on Ω, see above) with a support compact in Ω.

Definition 1.58. *A* seminorm $|\cdot|$ *on a linear space X is a real valued function satisfying Definition 1.48 with (H1) replaced by (H1)': $|x| \geq 0$, and $|x| = 0$* if *$x = 0$ (not iff $x = 0$).*

Introduce the seminorm[2] of the generalized derivative $D^k f$:

$$|f|_{m,p,\Omega} = \left(\sum_{|k|=m} (\|D^k f\|_p)^p \right)^{1/p}. \tag{1.51}$$

For $|k| = 0$ we have

$$|f|_{0,p,\Omega} = \left(\int_\Omega |f|^p d\Omega \right)^{1/p}. \tag{1.52}$$

Now we introduce the norm

$$\|f\|_{m,p,\Omega} = \left(\sum_{|k|=0}^{m} (|f|_{k,p})^p \right)^{1/p} \tag{1.53}$$

with

$$\|f\|_{m,\infty,\Omega} = \max_{|k| \leq m} |D^k f|_{0,\infty,\Omega}, \tag{1.54}$$

and define the *Sobolev space* $W^{m,p}$ to be the completion of $C_B^m(\Omega)$ in the norm $\|\cdot\|_{m,p}$.

We now formulate the *Poincaré inequality: Let Ω be a bounded domain in $\mathbb{R}^n$. There is a positive constant C, depending on Ω and p, such that*

$$|f|_{0,p,\Omega} \leq C |f|_{1,p,\Omega} \quad \text{for every } f \in W_c^{1,p}(\Omega). \tag{1.55}$$

[2] We must distinguish the seminorm of a function and its absolute value. For this we denote the seminorm by $|f|$ and its absolute value by $|f(x)|$.

We recall that $\mathcal{D}(\Omega)$ is the set of functions having continuous derivatives of all orders in Ω, and having compact support in Ω, i.e., their supports, which are closed, lie inside $\bar{\Omega}$. We define $W_0^{m,p}(\Omega)$ to be the completion of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{m,p}$:

$$W_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}. \tag{1.56}$$

For $p = 2$ we will use the notation:

$$W_0^{m,p}(\Omega) = H_0^m(\Omega). \tag{1.57}$$

$W_0^{m,p}(\Omega)$ is a subspace of $W^{m,p}(\Omega)$.

1.2.7 Some embedding theorems in a Sobolev space

An example of embedding theorem is given by Theorem 1.55. Recall that

$$\|f\|_{0,\infty,\Omega} = \sup_{x\in\Omega} |f(x)|$$

and

$$\|f\|_{m,\infty,\Omega} = \max_{|k|\le m} \sup_{x\in\Omega} |D^k f(x)|.$$

The spaces $C_B(\Omega)$, $C_c(\Omega)$, and $C(\bar{\Omega})$ are subsets of $C(\Omega)$. In fact,

$$C(\bar{\Omega}) \subset C_B(\Omega) \subset C(\Omega)$$

while

$$C^m(\bar{\Omega}) \subset C_B^m(\Omega) \subset C(\Omega).$$

The spaces $C(\bar{\Omega})$ and $C^m(\bar{\Omega})$ are Banach spaces. $C^{m+1}(\bar{\Omega})$ is a subspace of $C^m(\bar{\Omega})$, and

$$\|f\|_{m,\infty,\Omega} = \max_{|k|\le m} \sup_{x\in\Omega} |D^k f(x)| \le \max_{|k|\le m+1} \sup_{x\in\Omega} |D^k f(x)| = \|f\|_{m+1,\infty,\Omega} \tag{1.58}$$

so that the operator from $X \equiv C^{m+1}(\bar{\Omega})$ to $Y \equiv C^m(\bar{\Omega})$ is bounded. Thus,

$$C^{m+1}(\bar{\Omega}) \longrightarrow C^m(\bar{\Omega}). \tag{1.59}$$

Recall that $C(\bar{\Omega})$ is the set of functions that are bounded and uniformly continuous on Ω.

Definition 1.59. *The function $f(x)$ is uniformly continuous on Ω if, for given $\varepsilon > 0$, we can find $\delta > 0$ such that if $x, y \in \Omega$ and $\|x - y\| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

Definition 1.60. *Let $0 < \lambda \le 1$. $C^{0,\lambda}$ is the subspace of $C(\bar{\Omega})$ consisting of those functions which satisfy the* Hölder condition*: there is a constant K (depending on f) such that*

$$|f(x) - f(y)| \le K\|x - y\|^{\lambda}, \quad x, y \in \bar{\Omega}. \tag{1.60}$$

Such functions are said to be Hölder-continuous *or* Lipschitz-continuous *if* $\lambda = 1$.

Theorem 1.61. $C^{0,\lambda}(\bar{\Omega})$ *is the Banach space with the norm*

$$\|f(x)\|_{0,\lambda,\Omega} = \|f\|_{0,\infty,\Omega} + \sup_{x,y\in\Omega,\, x\neq y} \frac{|f(x)-f(y)|}{\|x-y\|^{\lambda}}. \tag{1.61}$$

The inequality $\|f\|_{0,\infty} \leq \|f\|_{0,\lambda}$ which follows from the definition (1.61) gives the continuous embedding

$$C^{0,\lambda}(\bar{\Omega}) \longrightarrow C(\bar{\Omega}). \tag{1.62}$$

Definition 1.62. *Let* $0 < \lambda \leq 1$. $C^{m,\lambda}(\bar{\Omega})$ *is a subspace of* $C^m(\bar{\Omega})$ *consisting of those functions with derivatives* $D^k f(x)$, $|k| \leq m$, *satisfying the Hölder condition, that is, there is a constant* K *depending of* f *such that*

$$|D^k f(x) - D^k f(y)| \leq K\|x-y\|^{\lambda}, \quad x, y \in \bar{\Omega},\ |k| \leq m.$$

We now introduce the notation

$$H_{k,\lambda,\Omega}(f) = \sup_{x,y\in\Omega,\, x\neq y} \frac{|D^k f(x) - D^k f(y)|}{\|x-y\|^{\lambda}} \tag{1.63}$$

and the definition

$$\|f\|_{m,\lambda,\Omega} = \|f\|_{m,\infty,\Omega} + \max_{|k|\leq m} H_{k,\lambda,\Omega}(f). \tag{1.64}$$

It can be proved that

$$\|f\|_{m,\infty,\Omega} \leq \|f\|_{m,\lambda,\Omega} \tag{1.65}$$

and that the space $C^{m,\lambda}(\bar{\Omega})$ is a Banach space. Then, the inequality (1.65) gives the embedding

$$C^{m,\lambda}(\bar{\Omega}) \longrightarrow C^m(\bar{\Omega}). \tag{1.66}$$

Theorem 1.63. *If* $0 < \nu < \lambda \leq 1$, $|k| \leq m$, *then the following embedding holds:*

$$C^{m,\lambda}(\bar{\Omega}) \longrightarrow C^{m,\nu}(\bar{\Omega}). \tag{1.67}$$

Theorem 1.64 (Sobolev embedding theorem). *Let* Ω *be a domain in* $\mathbb{R}^n$, *let* m *be an nonnegative integer, and* $1 \leq p < \infty$:

- *If* $mp < n$ *and if* $p \leq q \leq np/(n-mp)$, *then*

$$W_0^{m,p}(\Omega) \longrightarrow L^q(\Omega). \tag{1.68}$$

- *If* $mp = n$ *and if* $p \leq q < \infty$, *then*

$$W_0^{m,p}(\Omega) \longrightarrow L^q(\Omega). \tag{1.69}$$

- *If* $p = 1$ *and* $m = n$*, then*

$$W_0^{n,1}(\Omega) \longrightarrow C_B(\Omega). \tag{1.70}$$

- *If* $mp > n$*, then*

$$W_0^{m,p}(\Omega) \longrightarrow C_B(\Omega). \tag{1.71}$$

- *If* $mp > n > (m-1)p$*,* $0 \le \lambda \le m - n/p$*, then*

$$W_0^{m,p}(\Omega) \longrightarrow C^{0,\lambda}(\bar{\Omega}). \tag{1.72}$$

- *If* $n = (m-1)p$*,* $0 \le \lambda < 1$*, then*

$$W_0^{m,p}(\Omega) \longrightarrow C^{0,\lambda}(\bar{\Omega}). \tag{1.73}$$

- *If* $n = (m-1)$*,* $p = 1$*,* $0 \le \lambda \le 1$*, then*

$$W_0^{m,p}(\Omega) \longrightarrow C^{0,\lambda}(\bar{\Omega}). \tag{1.74}$$

Definition 1.65. *The domain* $\Omega \subset \mathbb{R}^n$ *has the* Lipschitz property *if for each point* $x \in \partial\Omega$ *there is a neighborhood in which the boundary is the graph of a Lipschitz continuous function.*

More detailed description of the Lipschitz property of a boundary is the following [FK80].

Consider a bounded domain $\Omega \in \mathbb{R}^n$ and and a point $x \in \Sigma = \partial\Omega$. Denote by B the closed ball with the center x and radius r, and by $\tilde{B}$ the projection of the set $\Gamma = \partial\Omega \bigcap B$ onto the hyperplane $y_n = 0$ in a Cartesian coordinate system $(y_1, y_2, \ldots, u_{n-1}, y_n)$. We suppose that there exists such a coordinate system and positive numbers r and ε, for which

1. There exists a function $a = a(y_1, y_2, \ldots, y_{n-1}) \in C^{0,1}(\tilde{B})$ such that the boundary Γ is given by the equation

$$y_n = a(y_1, y_2, \ldots, y_{n-1}), \tag{1.75}$$

i.e.,

$$\Gamma = \{(y_1, y_2, \ldots, y_{n-1}, y_n) \mid (y_1, y_2, \ldots, y_{n-1}) \in \tilde{B}, \\ y_n = a(y_1, y_2, \ldots, y_{n-1})\}. \tag{1.76}$$

2. The set

$$\Upsilon_{\text{int}} = \{y_n = a(y_1, y_2, \ldots, y_{n-1}, y_n) \mid (y_1, y_2, \ldots, y_{n-1}) \in \tilde{B}, \\ a(y_1, y_2, \ldots, y_{n-1}) - \varepsilon < y_n < a(y_1, y_2, \ldots, y_{n-1})\} \tag{1.77}$$

is in the domain Ω and the set

$$\Upsilon_{\text{ext}} = \{y_n = a(y_1, y_2, \ldots, y_{n-1}, y_n) \mid (y_1, y_2, \ldots, y_{n-1}) \in \tilde{B}, \\ a(y_1, y_2, \ldots, y_{n-1}) + \varepsilon < y_n < a(y_1, y_2, \ldots, y_{n-1})\} \tag{1.78}$$

is outside of the domain Ω.

An example of domain which does not satisfy the Lipschitz condition is given in [FK80, p. 49]. Note too that in reality we deal with a system of so-called *local cards* [Ada75] and corresponding local coordinate system. Denoting by "r" the number of a local card we will have the set of dependencies $y_n = a_r(y_1^r, y_2^r, \dots, y_{n-1}^r)$. For simplicity, below we keep the notation (1.76).

Theorem 1.66. *Let Ω be a domain in $\mathbb{R}^n$ having the Lipschitz property, and let m be a nonnegative integer, and $1 \le p < \infty$. Then, the embeddings (1.73) hold with $W_c^{m,p}$ replaced by $W^{m,p}$.*

We formulate also the essential results obtained by Sobolev and Kondrashov–Rellich for embedding operators as the theorem:

Theorem 1.67. *Let Ω be a domain in $\mathbb{R}^n$ having the Lipschitz property, and let m be a nonnegative integer, and $1 \le p < \infty$. Then,*

$$W^{m,p}(\Omega) \begin{cases} \hookrightarrow L^{p^*}(\Omega), & \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{m}{n},\ mp < n \\ \overset{c}{\hookrightarrow} L^q(\Omega), & 1 \le q < p^*,\ \textit{if } mp < n, \\ \overset{c}{\hookrightarrow} L^q(\Omega), & 1 \le q < \infty,\ \textit{if } mp = n, \\ \overset{c}{\hookrightarrow} C_0(\Omega), & \textit{if } mp > n, \end{cases} \tag{1.79}$$

where the symbol $\hookrightarrow$ denotes the continuity of the embedding, the symbol $\overset{c}{\hookrightarrow}$ the compactness of the embedding, and n the dimension of the space.

1.2.8 Dual (conjugate) spaces and weak convergence. Quotient space

Let X and Y be Banach spaces, and let A be a linear operator from X into Y (see Definition 1.45). The set of a linear operators on X is a linear space.

Theorem 1.68. *The space of the linear continuous operators on X is a Banach space.*

Definition 1.69. *If A is a* functional *then the space of the linear functionals on the space X is called a* dual *or* conjugate *to X. It will be denoted by X^* or $\mathcal{L}(X, \mathbb{R})$. We will denote a functional by $L(x) = \langle x, x^* \rangle$, $x^* \in X^*$.*

Since $X^* \equiv \mathcal{L}(X, \mathbb{R})$ is a linear space then we can introduce the space X^{**} of linear functionals on X^*. Due to Theorem 1.68 the space X^{**} is a Banach space. Note that $(l^p)^* = l^q$, $(L^p)^* = L^q$ where $(1/p) + (1/q) = 1$. Note also that $X \subset X^{**}$, in general.

Definition 1.70. *If $X^{**} = X$ then the Banach space X is called a* reflexive *Banach space.*

Definition 1.71. *Let X be a normed linear space. The sequence $\{x_k\} \subset X$ is said to be a* weak Cauchy sequence *if, for every linear functional $L(x) \in X^*$, the sequence $\{L(x_k)\}$ is a Cauchy sequence in $\mathbb{R}$. The sequence $\{x_k\} \subset X$ is said to* converge weakly *to $x_0 \in X$ if, for every linear continuous functional $L(x)$ on X,*

$$\lim_{k\to\infty} L(x_k) = L(x).$$

The convergence with respect to a norm, i.e., $\|x_k - x_0\| \to 0$, is said to be strong *convergence.*

We will use the previous notation $x_k \to x_0$ for strong convergence, $x_k \xrightarrow{w} x_0$ for weak convergence.

Theorem 1.72. *Let X be a normed linear space. If the sequence $\{x_k\} \subset X$ is a strong Cauchy sequence, then it is a weak Cauchy sequence. The opposite statement is not true (in the general case).*

Using the definition of the weak convergence we can define the weak closedness, weak completeness, and weak compactness of a set or space replacing "convergence" in the above definitions by "weak convergence." Note that sometimes the definition of $*$-weak convergence is used.

Definition 1.73. *Let X be a normed linear space, and let X^* be its dual space. The sequence $\{L_k\} \subset X^*$ is said to be a* $*$-weak Cauchy sequence *if, for every element $x \in X$, the sequence $\{L_k(x)\}$ is a Cauchy sequence in $\mathbb{R}$. The sequence $\{L_k\} \subset X^*$ is said to* $*$-converge weakly *to $L \in X^*$ if, for every element $x \in X$,*

$$\lim_{k\to\infty} L_k(x) = L(x).$$

We will use the notation $x_k \xrightarrow{w^*} x_0$ for $*$-weak convergence.

We now consider a Hilbert space H, with an inner product (u, v) for all $u, v \in H$.

Definition 1.74. *Let H be a Hilbert space and $M \subset H$ a linear subspace. The element $u \in H$ is said to be* orthogonal *to M if u is orthogonal to every $v \in M$, i.e., $(u, v) = 0$ for all $v \in M$. Two subspaces $M, N \subset H$ are said to be* mutually orthogonal *if $(u, v) = 0$ for all $u \in M$ and $v \in N$.*

We will write $M = N^\perp$, $N = M^\perp$.

Definition 1.75. *Let H be a Hilbert space and $M, N \subset H$ mutually orthogonal subspaces. We say that H has an* orthogonal decomposition *into M and N if any $u \in H$ can be uniquely represented in the form*

$$u = v + w, \quad v \in M,\ w \in N. \tag{1.80}$$

Theorem 1.76. *Let H be a Hilbert space and $M \subset H$ a closed subspace. Then, there is a closed subspace $N \subset H$, orthogonal to M, such that H has an orthogonal decomposition into M and N.*

Definition 1.77. *Let H be a Hilbert space and $M \subset H$ a closed subspace. The* projection operator *P on H onto M is defined by $Px = u$, where*

$$\|x - u\| = \inf_{y \in M} \|x - y\|.$$

Using Theorem 1.76, we can prove the *Riesz* representation theorem:

Theorem 1.78. *Let H be a Hilbert space, and $L(v)$ be a continuous linear functional on H. There is a unique $f \in H$ such that*

$$L(v) = (v, f) \equiv \langle v, f \rangle \quad \text{for every } v \in H \tag{1.81}$$

and $\|L\| = \|v\|$.

Since a Hilbert space is a normed space then the above definition of weak convergence is defined in such a space too.

Definition 1.79. *Let P_{k-1} be a set of the polynomials of power $\leq (k-1)$, and let $P \subset P_{k-1}$ be a linear subset. We denote by H^k/P* quotient space *to $H^k(\Omega)$ defined as the set of classes $\{\tilde{u}\}$ where a function $u \in \tilde{u}$ belongs to $H^k(\Omega)$, and*

$$u, v \in \tilde{u} \Leftrightarrow u - v \in P.$$

The norm in H^k/P is defined as follows:

$$\|\tilde{u}\|_{H^k/P} = \inf_{u \in \tilde{u}} \|u\|_{H^k(\Omega)}. \tag{1.82}$$

1.3 Bases and complete systems. Existence theorem

1.3.1 Bases and complete systems

Definition 1.80. *Let X be a normed linear space. A system of elements $g_1, g_2, \ldots \subset X$ is said to be a* basis *for X if any element $x \in X$ has a unique representation*

$$x = \sum_{k=1}^{\infty} \alpha_k g_k \tag{1.83}$$

with scalars α_k.

It follows from Definition 1.80 that if $x_m = \sum_{k=1}^{m} \alpha_k g_k$, then $\lim_{m\to\infty} \|x - x_m\| = 0$, and that a basis $g_1, g_2, \ldots$ is a linearly independent system since the equation

$$\sum_{k=1}^{\infty} \alpha_k g_k = 0 \tag{1.84}$$

has a unique solution $\alpha_1, \alpha_2, \ldots = 0$.

Definition 1.81. *Let X be a normed linear space. X is called* separable *if it contains a countable subset which is dense in X.*

Theorem 1.82. *If a normed linear space has a basis $\{g_k\}_{k=1}^{\infty}$, then it is separable. A dense set in it is a countable set of linear combinations of the form*

$$\sum_{k=1}^{\infty} c_k g_k$$

with $c_k \in \mathbb{R}$.

Definition 1.83. *Let X be a normed linear space. A countable system $g_1, g_2, \ldots \subset X$ is said to be* complete *in X if for any $x \in X$ and any $\varepsilon > 0$ there is a finite linear combination of the g_k such that*

$$\left\| x - \sum_{k=1}^{m} \alpha_k g_k \right\| \leq \varepsilon. \tag{1.85}$$

Definition 1.84. *Let H be a Hilbert space. A system of elements $\{g_k\} \subset H$ is said to be* orthonormal *if, for all integer m, l,*

$$(g_m, g_l) = \delta_{ml} = \begin{cases} 1, & \text{if } m = l, \\ 0, & \text{if } m \neq l. \end{cases}$$

Theorem 1.85. *Let H be a Hilbert space. If H has a complete orthonormal system $\{g_k\} \subset H$, then it is a basis for H. Any element $f \in H$ has a unique representation*

$$f = \sum_{k=1}^{\infty} \alpha_k g_k \tag{1.86}$$

called a Fourier series *for f. The numbers $\alpha_k = (f, g_k)$ are called the* Fourier coefficients *of f.*

Definition 1.86. *Let H be a Hilbert space. A system $\{g_k\} \subset H$ is said to be* closed *in H if the system of equations*

$$(f, g_k) = 0, \quad k = 1, 2, \ldots, \tag{1.87}$$

implies $f = 0$.

Theorem 1.87. *An orthonormal system in a Hilbert space H is closed iff it is complete.*

1.3.2 Existence of a solution of a set of linear equations

Let H be a Hilbert space. Consider the problem: find $u \in H$ such that the equation

$$(u, v) + L(v) = 0, \quad \forall v \in H \tag{1.88}$$

holds. As earlier, $L(v)$ is a linear functional on H.

Theorem 1.88. *Let $L(v)$ be a continuous linear functional on a Hilbert space H. There is a unique element $u \in H$ which satisfies (1.88).*

Definition 1.89. *A set K in a linear space is said to be* convex *if, for any two elements $x, y \in K$, each element $\lambda x + (1 - \lambda)y$ with $0 \leq \lambda \leq 1$ is also in K.*

Definition 1.90. *A normed linear space is called* strictly normed *if the equality*

$$\|x + y\| = \|x\| + \|y\|, \quad x \neq 0,$$

implies $y = \lambda x$, $\lambda \in \mathbb{R}$.

Theorem 1.91. *Let X be a strictly normed linear space, let $x \in X$, and let $K \subset X$ be a closed convex set. There is no more than one $y \in K$ which minimizes the functional $F(y) = \|x - y\|$ on K.*

This theorem can be generalized for the minimization problem for a convex functional.

Definition 1.92. *A functional $J(v)$ is called convex in the space V if, for two arbitrary elements v_1, v_2 and any number $t \in [0, 1]$, the inequality*

$$J((1 - t)v_1 + tv_2) \leq (1 - t)J(v_1) + tJ(v_2) \tag{1.89}$$

holds. If the inequality is strict for all $v_1 \neq v_2$ and all t such that $0 < t < 1$ the functional $J(u)$ is called strictly convex.

Definition 1.93. *Let H be a Hilbert space. A* bilinear form *$a(u, v)$ on H is a functional $a : H \times H \to \mathbb{R}$ which is a linear on u and v.*

This form is said to be continuous *if*

$$|a(u, v)| \leq M\|u\|\|v\|, \quad \forall u, v \in H,$$

where the positive constant M does not depend on u, v.

The Riesz representation theorem (see Theorem 1.78) defines an operator A by

$$a(u, v) = (Au, v), \quad \forall v \in H, \tag{1.90}$$

which is a continuous linear operator, i.e., $A \in \mathcal{L}(H, H)$.

The Riesz representation theorem defines another operator A^* by

$$a(u, v) = (u, A^* v), \quad \forall u \in H, \tag{1.91}$$

called the *adjoint* or *dual* to A. If $A^* = A$, then the operator A is said to be *self-adjoint*. Note that the most of the BVPs of the mechanics of solids correspond to self-adjoint operators.

Definition 1.94. *A bilinear form on H is said to be positive definite (or coercive or H-elliptic) if*

$$a(v, v) \geq \alpha \|v\|^2, \quad \forall v \in H, \tag{1.92}$$

where the positive constant α does not depend on v.

Theorem 1.95 (Lax–Milgram theorem). *Let $a(u, v)$ be a bilinear continuous coercive form on a Hilbert space H. Then this form defines an operator $A \in \mathcal{L}(H, H)$ which has an inverse $A^{-1} \in \mathcal{L}(H, H)$.*

An existence theorem for the solution of a functional minimization problems on a set $K \subset$V will be given in Section 3.3.

1.4 Trace theorem

Theorem 1.96. *Suppose that the boundary of the domain $\Omega \subset \mathbb{R}^n$ has the Lipschitz property (see Definition 1.65). Then there exists one and only one continuous linear operator* Tr *which defines for every function $v \in W^{1,p}$ a function* $\operatorname{Tr} u \in L^p(\Sigma)$ *satisfying*

$$\operatorname{Tr} u = u|_\Sigma \tag{1.93}$$

for all $u \in C^\infty(\bar{\Omega})$.

In the proof, we use the Lipshitz property of the boundary which permits to reduce a function $f(y_1, y_2, \ldots, y_{n-1}, y_n)$ in domain $\Gamma = \partial\Omega \bigcap B$ to the function $f(y_1, y_2, \ldots, y_{n-1}, a(y_1, y_2, \ldots, y_{n-1}))$ (see (1.76)), definition of the integral

$$\int_\Gamma f(y_1, y_2, \ldots, y_{n-1}, y_n)\, d\Sigma = \int_{\tilde{B}} f(y_1, \ldots, y_{n-1}, a(y_1, y_2, \ldots, y_{n-1}) \cdot \left[1 + \sum_{i=1}^{n-1} \left(\frac{\partial a}{\partial y_i}(y_1, \ldots, y_{n-1}) \right)^2 \right]^{1/2} dy_1 \ldots dy_{n-1}, \tag{1.94}$$

and definition of the integral over whole the surface Σ as a combination of integrals of type (1.94), using an appropriate cover of the surface Σ by the set of Cartesian coordinate system $(y_1, y_2, \ldots, y_{n-1}, y_n)$ (see, e.g., [Neč67], for an elementary proof see [NH80, p. 73]). In the definition (1.94) we use the known rule for the calculation of an integral over a surface and the formulae for the components ν_i of the vector ν of the outward drawn normal to the surface Γ:

$$\nu_i = \left[1 + \sum_{i=1}^{n-1}\left(\frac{\partial a}{\partial y_i}(y_1,\dots,y_{n-1})\right)^2\right]^{-1/2} \frac{\partial a}{\partial y_i}, \quad i = 1, 2, \dots (n-1),$$

$$\nu_n = \left[1 + \sum_{i=1}^{n-1}\left(\frac{\partial a}{\partial y_i}(y_1,\dots,y_{n-1})\right)^2\right]^{-1/2}. \tag{1.95}$$

Consider the most important applications in the case $k = 2$ and a space $W^{2,p}(\Omega)$. For all α, $|\alpha| = 1$, we have

$$D^\alpha u \in W^{1,p}(\Omega), \tag{1.96}$$

where the derivatives are the generalized ones. Then we can define the trace of the first generalized derivative of a function $u \in W^{2,p}(\Omega)$ by the formula (1.93), taking into account the fact that the main hypothesis of Theorem 1.96 holds.

From Theorem 1.96 it follows that

$$\|u\|_{L^p(\Sigma)} \le \gamma \|u\|_{1,p,\Omega}, \quad \forall u \in C^\infty(\bar{\Omega}), \tag{1.97}$$

where $\gamma = \text{const}$. The inequality (1.97) can be extended to the whole $W^{1,p}(\Omega)$.

Note that the trace of the mth derivative cannot be defined because there does not exist a continuous mapping T from $L^p(\Omega)$ into $L^p(\Sigma)$ such that $Tf = f|_\Sigma$ for a function $f \in C^0(\bar{\Omega})$.

Such a definition is useful for a theoretical investigation. It can be introduced if we suppose that the domain $\Omega \subset \mathbb{R}^n$ belongs to the class C^∞, i.e., the function a in the definition (1.75) is in a space $C^\infty(\Omega)$ (see Definition 1.25).

Let s be a real number, which satisfies $0 < s < 1$. Define a space $H^s(\Sigma)$ as the space of the functions, with

$$u \in H^s(\Sigma) \Longleftrightarrow \begin{cases} u \in L^2(\Sigma), \\ \displaystyle\int_{\Sigma\otimes\Sigma} \frac{|u(x) - u(y)|^2}{\|x - y\|^{n-1+2s}}\, d\Sigma_x\, d\Sigma_y < +\infty, \end{cases}$$

where $\|x - y\|$ is the Euclidean norm in $\mathbb{R}^n$, $\Sigma \otimes \Sigma$ is the set of all the pairs (x, y), $x \in \Sigma$, $y \in \Sigma$.

Equipped by the inner product

$$(u, v)_{H^s(\Sigma)} = \int_{\Sigma\otimes\Sigma} \frac{(u(x) - u(y))(v(x) - v(y))}{\|x - y\|^{n-1+2s}}\, d\Sigma_x\, d\Sigma_y + \int_\Sigma uv d\Sigma < +\infty, \tag{1.98}$$

the space $H^s(\Sigma)$ is a Hilbert space.

Note that an alternative definition of the space $H^s(\Omega)$ for noninteger and nonpositive s can be introduced with the Fourier transform (see [LM73] or [DL72, p. 41]).

Let q be a real number such that $q = 2 + \frac{2}{n-2}$ for $n > 2$ and q have an arbitrary value for $n = 2$. Then, we can define a trace operator $\mathrm{Tr} \in \mathcal{L}(H^1(\Omega); L^q(\Sigma))$ which have as range of values the space $H^{1/2}(\Sigma)$, i.e.,

1. For all $v \in H^1(\Omega)$, $\mathrm{Tr}\, v \in H^{1/2}(\Sigma)$
2. For all $g \in H^{1/2}(\Sigma)$, there exists $v \in H^1(\Omega)$ such that $\mathrm{Tr}\, v = g$

By definition, the space $H^{-1/2}(\Sigma)$ is the space dual to $H^{1/2}(\Sigma)$, i.e., it is the space of the linear functionals on $H^{1/2}(\Sigma)$.

The most important property of the space $H^{1/2}(\Sigma)$ follows from [DL72, Theorem 4.2, p. 43]:

Theorem 1.97. *If the hypotheses of Theorem 1.96 holds then the map* $v \to \mathrm{Tr}\, v$ *is linear continuous and surjective from* $H^1(\Omega)$ *to* $H^{1/2}(\Omega)$.

It follows from this theorem that

$$\|v\|_{H^{1/2}(\Sigma)} \leq \mathrm{const}\|v\|_{H^1(\Omega)}. \tag{1.99}$$

An analogous theorem holds in the space $H^{m-1/2}(\Sigma)$:

$$\|v\|_{H^{m-1/2}(\Sigma)} \leq \mathrm{const}\|v\|_{H^m(\Omega)}, \quad m \geq 1. \tag{1.100}$$

Proofs of such theorems are in [LM73]. The inequalities (1.99) and (1.100) will be used in Section 4.3.

1.5 Laws of thermodynamics

1.5.1 Basic definitions

In this chapter we use the monographs [Ger73, Leo50, Sed97, PG05].

Consider a mass (material) system which occupies the domain $\Omega \in \mathbb{R}^3$ with the boundary $\Sigma = \partial\Omega$. The term "mass" (or "material") means that in this domain the different physical processes take place, e.g., heat conduction from one material point to an another, mechanical deformation, and transformation of the mechanical work into heat energy. We use the notation Ω both for a domain in $\mathbb{R}^3$ and for the corresponding physical system in this domain.

Suppose that the state of this system is defined uniquely by finite number of parameters (variables) $\{\pi_0, \pi_1, \ldots, \pi_m\} = \{\pi_i\}_{i=0}^m \equiv \mathcal{E}$ called the *state parameters*. In the mechanics of solids these parameters (material density, temperature, deformation, etc.) depend, in general, on the spatial coordinates. In thermodynamics the *localization principle* is used with which we investigate an infinitely small neighborhood of a given point in a *heterogeneous* (nonhomogeneous) system and suppose that any state of such a neighborhood is homogeneous.

Each function which characterizes the state of the considered system and depends on the state parameters $\{\pi_0, \pi_1, \ldots, \pi_m\}$ only is a *state function.*

The domain of all the possible values of the state parameters is denoted by $\mathcal{V}$, being a connected variety. In thermodynamics we compare an initial state $\mathcal{E}_1$, a final state $\mathcal{E}_2$, and investigate a process $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ which connects the initial and final states. A process is represented by a curve in the variety $\mathcal{V}$. By supposition, this curve is continuous. Note that some physical processes are represented by a curve with a finite discontinuity. Later, we consider continuous curves only.

Suppose that a process $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ is accompanied by the *work* performed by the system Ω and by the *heat inflow* to this system. The work performed by the system is equal and opposite to the work performed on this system by the surrounding medium. We suppose that these notions are familiar to the reader. Also note that the heat inflow is related to the definition of *temperature.* The strict definition of the temperature, denoted here by $\pi_0 \equiv T$, was given by Caratheodory (see below). We denote the work for a process $\mathcal{F}$ by $A_{\mathcal{F}}$, and the corresponding heat inflow by $Q_{\mathcal{F}}$.

The laws of thermodynamics operate with the inflows δA, δQ corresponding to the infinitesimal increments $d\pi_i$, $i = 0, \ldots, m$, of the state parameters.

The mechanical work A for the process $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ can be represented as the sum [GPS02]:

$$A_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)} = \int_1^2 \delta A = \int_1^2 \sum_{i=1}^m F_i \, d\pi_i \equiv \int_1^2 \omega_A. \tag{1.101}$$

By supposition, the heat inflow Q has the analogous form:

$$Q_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)} = \int_1^2 \delta Q = \int_1^2 \sum_{i=1}^m G_i \, d\pi_i \equiv \int_1^2 \omega_Q. \tag{1.102}$$

The integrands in (1.101) and (1.102) are called *differential forms.* In the mathematical theory of thermodynamics we can postulate that there exist two differential forms ω_A and ω_Q defined on the variety $\mathcal{V}$ (see, e.g., [Ger73, p. 337]):

$$\omega_A = \sum_{i=1}^m F_i d\pi_i, \quad \omega_Q = \sum_{i=1}^m G_i d\pi_i. \tag{1.103}$$

The quantities F_i, G_i are called the *fluxes.* An equation which relate a flux and the state parameters is a *governing equation* (or *complementary law* [Ger73] or *governing relation* [PG05]). We make this definition precise in Chapter 6.

Note that in this section we develop the theory concerning the mechanical work and heat inflow only. This theory can be generalized to physical processes which are accompanied by energy inflow δQ^* different from mechanical work and heat inflow. An example concerning the adhesion phenomena will be considered in Section 6.4.

A process $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ is called a *cyclic* process (or a *cycle*) if $\mathcal{E}_1 \equiv \mathcal{E}_2$. Denote by

$$\mathcal{F}(\mathcal{E}_1, \mathcal{E}_3) = \mathcal{F}(\mathcal{E}_1, \mathcal{E}_2) \bigcirc \mathcal{F}(\mathcal{E}_2, \mathcal{E}_3) \tag{1.104}$$

the *complex process* which is a *composition* of two processes $\mathcal{F}(\mathcal{E}_1, \mathcal{E}_2)$ and $\mathcal{F}(\mathcal{E}_2, \mathcal{E}_3)$, i.e., the final state of the first process is the initial state of the second process.

An *additivity axiom* is formulated as follows: for a complex process defined by (1.104) we have the equalities:

$$Q_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_3)} = Q_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)} + Q_{\mathcal{F}(\mathcal{E}_2,\mathcal{E}_3)}, \quad A_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_3)} = A_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)} + A_{\mathcal{F}(\mathcal{E}_2,\mathcal{E}_3)}. \tag{1.105}$$

A *wall* is introduced as a surface in physical space which separates a system from the external medium, or separates a subsystem Ω' from the system Ω. A wall is called *adiabatic* if the heat inflow through the wall is zero.

Definition 1.98. *A process in a system Ω is called* adiabatic *if it proceeds without any heat inflow, i.e., the system Ω is inside an adiabatic wall. An adiabatic process will be denoted by $\mathcal{A}$.*

Definition 1.99. *A system is* closed *if it is in a closed wall (in a shell), and the work and heat inflow through this wall are zero. A system is* adiabatically isolated *if it is in a closed wall, and heat inflow through this wall is zero.*

Let $\{\mathcal{F}\}$ be the set of all the possible processes, and $\{\mathcal{A}\}$ be the set of the adiabatic processes, i.e., for all $\mathcal{F} \in \{\mathcal{A}\}$, $Q_{\mathcal{F}} = 0$.

Denote by $\mathcal{N}(\mathcal{E}_0)$ the set of final states of an adiabatic process with initial state $\mathcal{E}_0$, and denote by $\mathcal{T}(\mathcal{E}_0)$ the set of states which are states of an adiabatic process with the final state $\mathcal{E}_0$, see Figure 1.1. An element of $\mathcal{N}(\mathcal{E}_0)$ is $1_N, 2_N, \ldots$, an element of $\mathcal{T}(\mathcal{E}_0)$ is $1_T, 2_T, \ldots$.

Axiom on the set $\{\mathcal{A}\}$ is

$$\mathcal{N}(\mathcal{E}_0) \cup \mathcal{T}(\mathcal{E}_0) = \mathcal{V}. \tag{1.106}$$

This axiom states that any two states from $\mathcal{V}$ can be connected by an adiabatic process. In general, it is impossible to take any such state as an initial state – this statement follows from the second law (see later).

In general, the direction of a process can play an important role. There exists a subset $\mathcal{R} \subset \mathcal{F}$ of so-called *reversible processes* where the direction is insignificant, i.e., the initial and final states have the same status. If in the reversible process we move from the state $\mathcal{E}_2$ to the state $\mathcal{E}_1$ the work and heat inflow change signs.

An example of an irreversible process is the temperature equalization of a closed system.

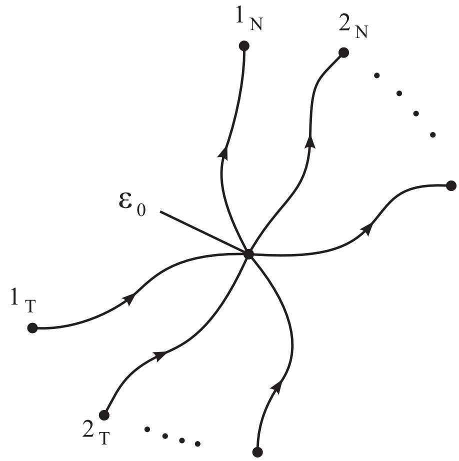

Fig. 1.1. The sets $\mathcal{N}(\mathcal{E}_0)$ and $\mathcal{T}(\mathcal{E}_0)$

1.5.2 First law of thermodynamics

Formulation of the first principle

Consider an adiabatic isolated system. The *first law of thermodynamics* for an adiabatic isolated system is formulated as follows: *In an adiabatic isolated system (i.e., $\delta Q = 0$) the work $A_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)}$ does not depend on the process $\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)$, but depends on the initial state $\mathcal{E}_1$ and the final state $\mathcal{E}_2$.*

It follows from this statement that we cannot get a positive work from a cyclic process, i.e., it is impossible to manufacture a *perpetuum mobile of the first kind.* An increment of mechanical work has the form (1.101).

Recall that π_0 is the temperature. The coefficients F_i are the generalized forces depending on the generalized coordinates π_i, $i = 1, 2, \ldots, m$, and on the temperature $\pi_0 = T$. Note that the work of the electromagnetic field and the internal stresses in a deformed solid (see Sections 3.2 and 3.3) have the analogous form.

The equation (1.101) and the first law of thermodynamics imply that the quantity $\delta A = \sum_{i=1}^{m} F_i \delta\pi_i$ is the total differential $-\delta E$ of the state function E called the *energy* (*total energy*) of the system Ω. So,

$$A_{\mathcal{F}(\mathcal{E}_1,\mathcal{E}_2)} = E_1 - E_2, \tag{1.107}$$

i.e., $E = -A + \text{const}$.

Note that the energy E is decomposed into three terms: kinetic, potential, and internal energy:

$$E = E_k + E_p + E_i, \tag{1.108}$$

where the kinetic energy E_k depends on the velocities of the movement of the system as a whole or on its macroscopic parts. The term macroscopic means, in particular, that we do not take into account the microstructure of the solids, i.e., we neglect the energy of the molecular and atomic motion. Note that the mean value of the energy of the molecular and atomic motion is the heat energy [Zie63]. Examples of the potential energy calculation will be given in Chapters 2 and 3. The remainder $E - E_k - E_p$ is the internal energy E_i.

Consider now an arbitrary thermodynamical system, i.e., the heat inflow is not zero. Define the energy (total energy) increment of the system Ω as the sum

$$\delta E = \delta A + \delta Q. \tag{1.109}$$

The first principle for this system is formulated as follows: *The increment of E in any cyclic process is zero.*

It follows from this principle that the increment δE is the total differential dE, because the increment of E in a cyclic process is the integral

$$E_\Gamma = \oint_\Gamma \delta E \, ds, \tag{1.110}$$

where Γ is a closed curve in the space of state parameters which corresponds to the cyclic process.

We emphasize that, in general, the influx of heat δQ is not a total differential.

Equilibrium state and the definition of temperature

Definition 1.100. *A* thermodynamic equilibrium *is a state in which the state parameters* $\{\pi_i\}_{i=0}^m$ *do not depend on time.*

Definition 1.101. *An infinitesimally slow process is called a quasi-static processes if it is a sequence of thermodynamic equilibrium states.*

Quasi-static processes are reversible. This statement follows from Definition 1.100, the definition of the thermodynamic equilibrium. If we change the direction of the change of external parameters and temperature in a quasi-static process, we get a sequence of equilibrium states inverse to that in the initial process. Note that, in general, the converse statement does not hold.

To give a rigorous definition of the temperature, we consider the particular case of a thermodynamic system Ω consisting of three subsystems Ω_1, Ω_2, Ω_3, and formulate the axioms following from the physical observations:

1. Let the system $\Omega_{12} = \Omega_1 \cup \Omega_2$ and system $\Omega_{23} = \Omega_2 \cup \Omega_3$ be in thermodynamic equilibrium. Suppose that there are no adiabatic walls between the subsystems. Then, the system $\Omega_{13} = \Omega_1 \cup \Omega_3$ is in thermodynamic equilibrium. This statement is known as the *transitivity* of thermodynamic equilibrium and can be generalized to any number of subsystems.
2. Distribution of the energy E of the system Ω over its subsystems is unique.
3. If the energy E increases, then the energies of the subsystems also increase.

Suppose that the equilibrium state of the subsystem Ω_{12} depends on two parameters π_1 and π_2 only. We will see that all the statements and transformations can be generalized to an arbitrary number of state parameters.

The energy of a subsystem can be considered as a state parameter of the whole system and *vice versa*. Then, in the thermodynamic equilibrium the energies E_1 and E_2 of subsystems Ω_1 and Ω_2 depend on π_1, π_2 and on the total energy $E = E_1 + E_2$, i.e.,

$$E_1 = f_1(\pi_1, \pi_2, E), \tag{1.111}$$

$$E_2 = f_2(\pi_1, \pi_2, E). \tag{1.112}$$

Additivity of the energy gives the equation

$$E_1 + E_2 = E. \tag{1.113}$$

Consider the relations (1.111)–(1.113) as a system of three nonlinear equations in the five variable π_1, π_2, E_1, E_2, and E. We can find a solution of this system as, e.g.,

$$\pi_2 = \pi_2(\pi_1, E_1), \quad E_2 = E_2(\pi_1, E_1), \quad E = \Phi_1(\pi_1, E_1) \tag{1.114}$$

or

$$\pi_1 = \pi_1(\pi_2, E_2), \quad E_1 = E_1(\pi_2, E_2), \quad E = \Phi_2(\pi_2, E_2). \tag{1.115}$$

Equating the two expressions for the energy E, we obtain

$$\Phi_1(\pi_1, E_1) = \Phi_2(\pi_2, E_2). \tag{1.116}$$

An analogous argument leads to the equation

$$\Psi_2(\pi_2, E_2) = \Psi_3(\pi_3, E_3) \tag{1.117}$$

for the system $\Omega_{23} = \Omega_2 \cup \Omega_3$, and to the equation

$$\Lambda_3(\pi_3, E_3) = \Lambda_1(\pi_1, E_1) \tag{1.118}$$

for the system $\Omega_{31} = \Omega_3 \cup \Omega_1$.

Let the systems Ω_{12} and Ω_{23} be in equilibrium states. Then, the system Ω_{31} is in equilibrium too due to the transitivity property. So, the equality (1.118) must follow from the equalities (1.116)–(1.117).

In particular, if we find the variable E_2 from equation (1.117) as $E_2 = E_2(\pi_2, \pi_3, E_3)$ and substitute this expression in (1.116), we conclude that $\Phi_2(\pi_2, E_2(\pi_2, \pi_3, E_3))$ does not depend on the variable π_2, i.e.,

$$\frac{\partial \Phi_2}{\partial \pi_2} + \frac{\partial \Phi_2}{\partial E_2}\frac{\partial E_2}{\partial \pi_2} = 0. \tag{1.119}$$

The same reasoning with the equation (1.117) (instead of (1.116)) leads to the equation

$$\frac{\partial \Psi_2}{\partial \pi_2} + \frac{\partial \Psi_2}{\partial E_2}\frac{\partial E_2}{\partial \pi_2} = 0. \tag{1.120}$$

From the equations (1.119)–(1.120) we obtain

$$\frac{\partial \Phi_2}{\partial \pi_2}\frac{\partial \Psi_2}{\partial E_2} - \frac{\partial \Phi_2}{\partial E_2}\frac{\partial \Psi_2}{\partial \pi_2} = 0. \tag{1.121}$$

This equation means that one of the two functions Φ_2 and Ψ_2 is a function of the other:

$$\Phi_2 = f(\Psi_2). \tag{1.122}$$

It follows from this result that the equation (1.116) takes the form

$$\Phi_2(\pi_2, E_2) = \Phi_3(\pi_3, E_3), \tag{1.123}$$

and that the equation (1.117) takes the form

$$\Phi_3(\pi_3, E_3) = \Phi_1(\pi_1, E_1). \tag{1.124}$$

The equalities (1.123)–(1.124) means that, *for any system, there exists a function of its state parameters and its energy which has the same value for all the systems in thermodynamic equilibrium after their union.*

Definition 1.102. *The functions $\Phi_1, \Phi_2, \Phi_3, \ldots$ are called the temperatures of the systems $\Omega_1, \Omega_2, \Omega_3, \ldots$.*

The temperature is not defined uniquely, because we can choose a function Θ which is the same for all the subsystems, and define a temperature by

$$\phi_i(\pi_i, E_i) = \Theta(\Phi_i(\pi_i, E_i)), \quad i = 1, 2, \ldots, m. \tag{1.125}$$

This statement can be formulated as follows: *there exist different scales of temperature.*

The set of such scales can be restricted with the hypotheses that increasing temperature increases energy and the uniqueness of the energy distribution between the subsystems of the system (see above). Using these hypotheses we conclude that the temperature is a monotonic function of the heat inflow. After this we can define a scale of temperature using physical experiments, e.g., we can define the Celsius scale using experiments with water heating.

1.5.3 Second law of thermodynamics

General form of the second law and some consequences

The most general formulation of the second law of thermodynamics is the following: *No system can produce a positive work due to the cooling of some part or a subsystem of this system.* This is the Clausius form of the second law.

It follows from this statement that it is impossible to manufacture a *perpetuum mobile of the second kind.*

Definition 1.103. *A system with an identical temperature for all its subsystems is called an isothermal system.*

It follows from the second law of thermodynamics that *work cannot be positive for any cyclic isothermal process.*

Indeed, by definition, an isothermal process is a sequence of states with constant temperature supported with a thermostat. If the work as positive, then this work would be received due to the cooling of the thermostat. The second law states that such a phenomena is impossible.

Another consequence of the second law, which is taken sometimes as the initial formulation of this law (see, e.g., [Ger73, p. 340]), concerns adiabatic processes.

Theorem 1.104 (Caratheodory theorem). *There exists a state of an isothermal system which cannot be connected with a given state by an adiabatic process.*

Proof. Consider the isothermal process $\mathcal{F}_T(\mathcal{E}_1, \mathcal{E}_2)$. Let the heat influx Q_{12} in this process be Q_{12}, $Q_{12} > 0$, and the work be A_{12}. Suppose that there exists an adiabatic process $\mathcal{F}_A(\mathcal{E}_2, \mathcal{E}_1)$ which connects the states $\mathcal{E}_2$ and $\mathcal{E}_1$. The work for all the complex process is equal to Q_{12}. If we can go to any state by an adiabatic process, then we would get a positive work for the considered complex process due to cooling only. This result contradicts the second law, and the contradiction proves the Caratheodory theorem.

Entropy

The most important concept of thermodynamics is the existence of the state function called the *entropy.*

To prove the existence of the entropy using the second law of thermodynamics, we consider an abstract differential form

$$\omega_m = X_1 dx_1 + X_2 dx_2 + \ldots + X_m dx_m \qquad (1.126)$$

(see the example (1.103)), and define an *integrating factor* as a function $\mu(x_1, x_2, \ldots, x_m)$ such that the product $\mu\omega_m$ is equal to the total differential of some function $\sigma(x_1, x_2, \ldots, x_m)$, i.e.,

$$\mu\omega_m = \mu(X_1 dx_1 + X_2 dx_2 + \ldots + X_m dx_m) = d\sigma(x_1, x_2, \ldots, x_m). \qquad (1.127)$$

If $m = 2$, then an integrating factor always exists. If $m > 2$, then an integrating factor does not exist, in general, see, e.g., [NS60].

Consider firstly a reversible process. For such a process the following theorem holds [Leo50].

Theorem 1.105. *(i) The heat inflow δQ for any reversible process always has an integrating factor.*

(ii) In the set of all the integrating factors there exists a factor depending on the temperature only.

Proof. It follows from the first law of thermodynamics that the increment of the total energy

$$\delta E = \delta A + \delta Q \tag{1.128}$$

is total differential dE. Suppose that the forces F_i are potential, i.e.,

$$F_i = -\frac{\partial \Pi}{\partial \pi_i}, \quad i = 1, 2, \ldots, m. \tag{1.129}$$

Then the increment δA is a total differential, too.

After substitution (1.129) in (1.128), we obtain

$$\delta Q = dE - \sum_{i=1}^{m} F_i d\pi_i = \frac{\partial E}{\partial \pi_0} d\pi_0 + \sum_{i=1}^{m} \frac{\partial}{\partial \pi_i}(E - \Pi) d\pi_i. \tag{1.130}$$

Introduce the notations

$$\sigma = -\frac{\partial \Pi}{\partial \pi_0}, \quad G = E - \Pi. \tag{1.131}$$

Then

$$\delta Q = dG - \sigma d\pi_0. \tag{1.132}$$

It follows from (1.132) that the heat inflow is defined by the three variables $\{G, \sigma, \pi_0\}$.

We prove that in the set $\{G, \sigma, \pi_0\}$ there are only two independent variables. The proof can be performed by contradiction.

Suppose that all the three variable are independent. We demonstrate that any two states $\{G_1, \sigma_1, \pi_0^{(1)}\}$ and $\{G_2, \sigma_2, \pi_0^{(2)}\}$ correspond to an adiabatic process, i.e., any two points in the three-dimensional (3D) space $\{G, \sigma, \pi_0\}$ can be connected by an continuous curve satisfying the equation

$$\delta Q = 0 = dG - \sigma d\pi_0. \tag{1.133}$$

Indeed, let the points $\{G_1, \sigma_1, \pi_0^{(1)}\}$ and $\{G_2, \sigma_2, \pi_0^{(2)}\}$ be connected by a line

$$G = G(\pi_0), \quad \sigma = \sigma(\pi_0). \tag{1.134}$$

It follows from (1.133) that the function G is a solution of the Cauchy problem

$$\frac{dG}{d\pi_0} = \sigma(\pi_0), \quad G|_{\pi_0 = \pi_0^{(1)}} = G_1. \tag{1.135}$$

The solution of this problem is the relation between two sets of functions, G and σ:

$$G(\pi_0) = G_1 + \int_{\pi_0^{(1)}}^{\pi_0} \sigma(\tilde{\pi}_0)\, d\tilde{\pi}_0. \tag{1.136}$$

Choose a function $\sigma(\pi_0)$ depending on three constants, to satisfy the equations

$$\sigma(\pi_0^{(1)}) = \sigma_1, \quad \sigma(\pi_0^{(2)}) = \sigma_2, \tag{1.137}$$

$$G(\pi_0^{(2)}) = G_2 = G_1 + \int_{\pi_0^{(1)}}^{\pi_0^{(2)}} \sigma(\tilde{\pi}_0)\, d\tilde{\pi}_0. \tag{1.138}$$

where the constants σ_1 and σ_2 define states at the curve $(\sigma(\pi_0)$¡ and $G_2 = G_1$ is a constant in the Cauchy problem (1.135). Such a choice is always possible (e.g., for $\sigma(\pi_0)$ being a polynomial of degree not less than two). Then we obtain the contradiction to the second law, i.e., any two states can be joined by an adiabatic process.

Suppose now that there exists only one independent variable in the set $\{G, \sigma, \pi_0\}$. Let π_0 be this independent variable. Then

$$G = \phi_G(\pi_0), \quad \sigma = \phi_\sigma(\pi_0). \tag{1.139}$$

By supposition, we consider an adiabatic process, then

$$\frac{d\phi_G}{d\pi_0} = \phi_\sigma(\pi_0), \tag{1.140}$$

but in the general case this equality cannot be satisfied, and any adiabatic process is impossible.

So, we have two independent variables only. Suppose that

$$G = g(\sigma, \pi_0). \tag{1.141}$$

It follows from (1.132) that

$$\delta Q = \frac{\partial g(\sigma, \pi_0)}{\partial \sigma} d\sigma + \left[\frac{\partial g(\sigma, \pi_0)}{\partial \pi_0} - \sigma \right] d\pi_0. \tag{1.142}$$

Now we can apply the theorem on the existence of an integrating factor for any linear differential form in two independent variables [NS60] to the form (1.142). To demonstrate that this factor can depend on the parameter π_0 only, we transform equation (1.142) as follows.

Recall the additivity property for the energy E:

$$E = E_1 + E_2, \tag{1.143}$$

where E_1 and E_2 are the energies of any subsystems Ω_1, Ω_2 such that $\Omega_1 \cup \Omega_2 = \emptyset$. An analogous equation is valid for the function Π, and from the definition of the function σ we obtain

$$\sigma = \sigma_1 + \sigma_2 \tag{1.144}$$

and

$$G = E - \Pi = G_1 + G_2. \tag{1.145}$$

By definition

$$G = g(\sigma, \pi_0), \quad G_1 = g_1(\sigma_1, \pi_0), \quad G_2 = g_2(\sigma_2, \pi_0). \tag{1.146}$$

It follows from (1.144)–(1.146) that

$$g(\sigma_1 + \sigma_2, \pi_0) = g_1(\sigma_1, \pi_0) + g_2(\sigma_2, \pi_0). \tag{1.147}$$

Differentiate this equation with respect to the variables σ_1, σ_2. Denoting this derivatives by $'$, we obtain

$$g'(\sigma_1 + \sigma_2, \pi_0) = g_1'(\sigma_1, \pi_0), \quad g'(\sigma_1 + \sigma_2, \pi_0) = g_2'(\sigma_2, \pi_0), \tag{1.148}$$

i.e.,

$$g_1'(\sigma_1, \pi_0) = g_2'(\sigma_2, \pi_0). \tag{1.149}$$

Then the derivatives $g_1' = g_2'$ do not depend on σ, and

$$g'(\sigma_1 + \sigma_2, \pi_0) = g_1'(\sigma_1, \pi_0) = g_2'(\sigma_2, \pi_0) = \alpha(\pi_0), \tag{1.150}$$

where the function α depends on the temperature π_0 only, and this function is the same for all the subsystems of the considered system.

The general solutions of the differential equations (1.150) are given by

$$\begin{aligned} g_1(\sigma_1, \pi_0) &= \alpha(\pi_0)\sigma_1 + \beta_1(\pi_0), \\ g_2(\sigma_2, \pi_0) &= \alpha(\pi_0)\sigma_2 + \beta_2(\pi_0), \\ g(\sigma_1 + \sigma_2, \pi_0) \equiv g(\sigma, \pi_0) &= \alpha(\sigma, \pi_0)\sigma + \beta(\pi_0). \end{aligned} \tag{1.151}$$

Using the equation $g = g_1 + g_2$, we prove that

$$\alpha(\pi_0)(\pi_1 + \pi_2) + \beta(\pi_0) = \alpha(\pi_0)(\pi_1 + \pi_2) + \beta_1(\pi_0) + \beta_2(\pi_0) \tag{1.152}$$

and that

$$\beta(\pi_0) = \beta_1(\pi_0) + \beta_2(\pi_0). \tag{1.153}$$

It follows from these results that

$$\sigma(\pi_0) = (G - \beta)/\alpha. \tag{1.154}$$

Substituting (1.154) in (1.132), we obtain the linear differential form on two variables:

$$\delta Q = dG - \frac{G - \beta}{\alpha} d\pi_0, \tag{1.155}$$

which has an integrating factor. We denote this factor by $1/T$ and prove that if $T = T(\pi_0)$, then $[dG - \frac{G-\beta}{\alpha} d\pi_0]/T$ is the total differential of a function S.

Indeed, by definition of the integrating factor,

$$\frac{1}{T}\delta Q = dS = \left[dG - \frac{G - \beta}{\alpha} d\pi_0 \right] \frac{1}{T}. \tag{1.156}$$

Choose

$$\frac{dT}{T} = \frac{d\pi_0}{\alpha(\pi_0)}. \tag{1.157}$$

Then

$$\frac{dG}{T} - \left[\frac{G-\beta}{\alpha} d\pi_0\right]\frac{1}{T} = \frac{dG}{T} - \frac{G}{T^2}dT + \frac{\beta}{T^2}dT = d\left[\frac{G}{T} + \int \beta \frac{dT}{T^2}\right], \tag{1.158}$$

and

$$S = \frac{G}{T} + \int \beta \frac{dT}{T^2}. \tag{1.159}$$

Finally, we prove that $1/T$ is an integrating factor depending on the variable π_0 only, and that

$$\delta Q = TdS, \tag{1.160}$$

where the function $S = S(\pi_0, \pi_1, \ldots, \pi_m)$ is called the *entropy* of the system Ω.

Note that we considered reversible processes only. For such processes the equation

$$\oint dQ = \oint TdS = 0. \tag{1.161}$$

holds. This equation is the *Clausius–Duhem* equation.

Fundamental inequality for entropy

Consider a cyclic process $\mathcal{F}_C$ called a *Carnot cycle* (Carnot process). If the number of the state parameters is two (e.g., π_1, π_2), then the Carnot cycle is a closed curve in the plane (see Figure 1.2).

This curve consists of four steps:

Step $1 \to 2$: an isothermal process, with nonzero heat influx $Q = Q_{12}$, and temperature $T_2 = \text{const}$

Step $2 \to 3$: an adiabatic process, with zero heat influx $Q = 0$

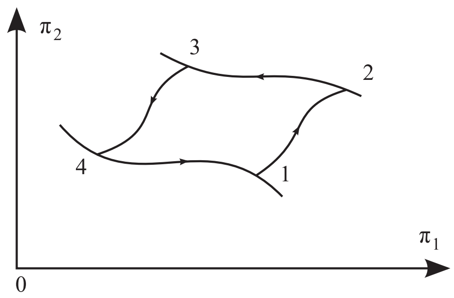

Fig. 1.2. Carnot cycle

Step 3 → 4: an isothermal process, with nonzero heat influx $Q = Q_{34}$, and temperature $T_1 = \text{const}$, $T_1 < T_2$
Step 4 → 1: an adiabatic process, with zero heat influx $Q = 0$

A Carnot cycle is closed. Then the work is

$$A_C = Q_{12} - Q_{34}. \tag{1.162}$$

The ratio

$$\eta = \frac{A_C}{Q_{12}} = 1 - \frac{Q_{34}}{Q_{12}} \tag{1.163}$$

is called the *efficiency* of the Carnot process. It follows from the second law that $A_C \leq Q_{12}$, also:

$$\eta \leq 1. \tag{1.164}$$

Theorem 1.106. *The efficiency of the reversible Carnot process is maximal with respect to any other Carnot processes.*

Proof. We denote by $A_C^{\text{rev}} = Q_{12}^{\text{rev}} - Q_{34}^{\text{rev}}$ the work for the reversible process, and by $A_C = Q_{12} - Q_{34}$ the work for an irreversible process. Let the considered system be a machine working with two sources of heat, the first has temperature T_2, and the second has temperature T_1. We suppose that $T_2 > T_1$. We consider an auxiliary system composed by two machines working together: the first is working as described above, and realizes a "direct Carnot process" (in general, nonreversible). The second machine realizes the reversible Carnot process (called "inverse process") for which, by supposition, this machine adds the heat Q_{12}^{rev} to the "hot" source with the temperature T_2 and takes the heat Q_{34}^{rev} from the "cold" source with the temperature $T_1 < T_2$. The total work of such composite machine for the Carnot cycle is calculated as

$$A^{\text{tot}} = (Q_{12} - Q_{34}) - (Q_{12}^{\text{rev}} - Q_{34}^{\text{rev}}). \tag{1.165}$$

Taking into account the first law, we conclude that the work of the reversible machine is calculated as

$$A^{\text{rev}} = Q_{12}^{\text{rev}} - Q_{34}^{\text{rev}}. \tag{1.166}$$

We suppose that the heat inflow Q_{12}^{rev} is equal to the heat inflow for the inverse process, i.e.,

$$Q_{12} = Q_{12}^{\text{rev}}. \tag{1.167}$$

Then

$$A^{\text{tot}} = Q_{34} - Q_{34}^{\text{rev}}. \tag{1.168}$$

The equality (1.168) means that the heat $Q_{34} - Q_{34}^{\text{rev}}$ is transformed completely into mechanical work, without any other change of the state of the system. Such a result contradicts to the second law, so that

$$\frac{Q_{34}^{\text{rev}}}{Q_{12}^{\text{rev}}} \leq \frac{Q_{34}}{Q_{12}}. \tag{1.169}$$

Note that the relation (1.169) is an equality if the initial process is reversible, too. In the proof of Theorem 1.106 we do not use any physical properties of the system. Thus, the efficiency depends on the temperatures T_1, T_2, i.e.,

$$\eta = \eta(T_1, T_2) \tag{1.170}$$

as the relation Q_1/Q_2.

We now prove that there exists a function $\varphi = \varphi(T)$ such that

$$\frac{Q_{34}}{Q_{12}} = \frac{\varphi(T_1)}{\varphi(T_2)}. \tag{1.171}$$

Let us introduce an intermediate source with the constant temperature T_{int} which absorbs and gives the same quantity of heat Q_{int} in two additional Carnot cycles $(T_1, Q_1; T_{\text{int}}, Q_{\text{int}})$ and $(T_{\text{int}}, Q_{\text{int}}; T_2, Q_2)$. The following relation holds:

$$\frac{Q_{34}}{Q_{12}} = \frac{Q_{34}}{Q_{\text{int}}} \cdot \frac{Q_{\text{int}}}{Q_{12}}. \tag{1.172}$$

Taking into account the arbitrariness in the choice of T_{int}, we conclude that the statement on the existence of the function $\varphi(T)$ and relation (1.171) are valid.

Let us choose a system of measurement units in which the dimensionality $[T] = \Theta$ appears as an independent unit of measurement, and demonstrate the *theorem on the absolute temperature*. This theorem will be proved only for a particular system – for a volume V_1 of ideal gas characterized by the pressure p, density ρ, and temperature T. The more general system can be investigated with the approach developed in Section 1.5.2.

The governing equation (the Clapeyron equation) for an ideal gas has the form $pV = \rho RT = R_0 T$, $R_0 = RM_0$, where R is the universal gas constant, M_0 is the molar mass.

Let V and T be the state parameters for an ideal gas

$$E = E(V, T) \tag{1.173}$$

and

$$\left(\frac{\partial E}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p. \tag{1.174}$$

By definition, the quantity

$$\left(\frac{\partial E}{\partial T}\right)_V = C_V \tag{1.175}$$

is the *specific heat at constant volume*, and

$$\left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = C_p \tag{1.176}$$

is the *specific heat at constant pressure*.

It follows from the definitions (1.175) and (1.176), and the formula (1.174) that

$$C_p - C_V = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p - \left(\frac{\partial E}{\partial T}\right)_V = \left[\left(\frac{\partial E}{\partial T}\right)_p + p\right]\left(\frac{\partial V}{\partial T}\right)_p . \tag{1.177}$$

We now consider an adiabatic process ($\delta Q = 0$). It follows from the definitions that for such a process

$$C_V dT + pdV = 0. \tag{1.178}$$

Taking into account equation $pV = R_0 T$, we obtain

$$C_V \frac{dT}{T} + R_0 \frac{dV}{V} = 0. \tag{1.179}$$

We now introduce the constant $\gamma = C_p/C_V$ called the *adiabatic exponent*. It follows from the equation (1.179) that for an adiabatic process

$$TV^{\gamma-1} = \text{const}, \quad pV^{\gamma} = \text{const}, \quad Tp^{1-\gamma}/\gamma = \text{const}. \tag{1.180}$$

Theorem 1.107 (Absolute temperature). *In the relation (1.171) we can choose* $\varphi(T) = T$.

Proof. We consider the following Carnot cycle:

Step $1 \to 2$: an isothermal extension from V_1 to V_2. It is known (see, e.g., [Ger73]) that in such a process the internal energy E is constant because $E = c_V T + \text{const}$, where c_V is the heat capacity per unit volume. Then all the heat influx is transformed into mechanical work, i.e.,

$$Q_{12} = \int_1^2 pdV = R_0 T_2 \ln \frac{V_2}{V_1}. \tag{1.181}$$

Step $2 \to 3$: an adiabatic extension from V_2 to V_3, with zero heat influx. For such a process $dQ = C_V dT + pdV = 0$ and

$$C_V(T_2 - T_1) = \int_2^3 p\, dV. \tag{1.182}$$

It follows from (1.180) that

$$T_2 V_2^{\gamma-1} = T_1 V_3^{\gamma-1}. \tag{1.183}$$

Step $3 \to 4$: an isothermal compression from V_3 to V_4. At this step we obtain

$$Q_1 = R_0 T_1 \ln \frac{V_4}{V_3}. \tag{1.184}$$

Step 4 → 1: an adiabatic compression from V_4 to V_1 with zero heat influx $Q = 0$, and

$$T_1 V_4^{\gamma-1} = T_2 V_1^{\gamma-1}. \tag{1.185}$$

We now compare (1.185) and (1.183). It follows from this comparison that

$$\frac{V_2}{V_1} = \frac{V_4}{V_3}. \tag{1.186}$$

Using (1.181) and (1.184), we finally obtain

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2}. \tag{1.187}$$

Theorem 1.108. *For any cycle the following inequality holds:*

$$\oint \frac{\delta Q}{T} \leq 0. \tag{1.188}$$

Proof. Consider first a cyclic process $\mathcal{F}_C$ in which the system interchanges heat with m thermostats with temperatures $T_1, T_2, \ldots, T_m$, and the corresponding heat quantities are $Q_1, Q_2, \ldots, Q_m$. Introduce an additional thermostat with temperature T_0, and consider m reversible Carnot cycles. In each of these cycles the system interchanges heat with two thermostats only – the first with the temperature T_i and the second with the temperature T_0. The heat inflow to the first thermostat is $Q_{(i)}$, and the heat decrease in the second thermostat is $Q_0^{(i)}$. For such the process the following equation holds:

$$\frac{Q_{(i)}}{Q_0^{(i)}} = \frac{T_i}{T_0}. \tag{1.189}$$

Let a complex process be the composition of the initial process $\mathcal{F}_C$ and m successive reversible Carnot cycles $\mathcal{F}_C^{(i)}$, $i = 1, 2, \ldots, m$. For this complex process the total heat increase Q_0 in the additional thermostat is

$$Q_0 = \sum_{i=1}^{m} Q_0^{(i)}. \tag{1.190}$$

Using the equation (1.189), we obtain

$$Q_0 = T_0 \sum_{i=1}^{m} \frac{Q_{(i)}}{T_i}. \tag{1.191}$$

Return now to the initial process. It follows from the second law that for this process

$$\frac{Q_0}{T_0} = \sum_{i=1}^{m} \frac{Q_{(i)}}{T_i} \leq 0. \tag{1.192}$$

Passage to the limit with $m \to \infty$ gives the inequality (1.188).

We recall that the integration in (1.188) is realized over a cycle, and emphasize that the equality take place for a reversible process.

We now prove the existence of entropy for the particular case of an ideal gas. Let a cycle in (1.188) be a reversible cycle. It follows from (1.188) that for such a process

$$\int_1^2 \frac{\delta Q^{\text{rev}}}{T} + \int_2^1 \frac{\delta Q^{\text{rev}}}{T} = 0, \tag{1.193}$$

i.e., the integral from a state "1" to a state "2" does not depend on the path. Thus, there exists a state function $S = S(p, V)$ such that

$$\frac{\delta Q^{\text{rev}}}{T} = dS, \tag{1.194}$$

which is the state function called the *entropy*. Using this definition, we obtain from (1.193)

$$\int_1^2 \frac{\delta Q^{\text{rev}}}{T} = -\int_2^1 \frac{\delta Q^{\text{rev}}}{T} = S_2 - S_1. \tag{1.195}$$

If one of the integrals in (1.193) is replaced by another one which corresponds to an irreversible process, then

$$\int_1^2 \frac{\delta Q}{T} + \int_2^1 \frac{\delta Q^{\text{rev}}}{T} \leq 0, \tag{1.196}$$

i.e.,

$$\int_1^2 \frac{\delta Q}{T} \leq S_2 - S_1. \tag{1.197}$$

This inequality means that in an adiabatically isolated system the entropy can only increase:

$$S_2 \geq S_1. \tag{1.198}$$

In an open system the entropy can decrease, but the total entropy of the system and its environments increases.

Using the definition (1.194), we conclude that

$$TdS \geq \delta Q. \tag{1.199}$$

This inequality is often used as a formulation of the second law.

We now introduce the *dissipation* or *function of dissipation* D^* by

$$TdS = \delta Q + D^*\, dt. \tag{1.200}$$

It follows from this definition that

$$D^* \geq 0. \tag{1.201}$$

Applications of the dissipation D^* will be given in Chapter 6.

Note in conclusion that heat conduction phenomena and related notions of temperature and entropy are a result of atomic and molecular motion. In 1877, Boltzmann obtained the dependence of the entropy on the thermodynamic probability W as (see, e.g., [PG05]):

$$S = k \ln W. \tag{1.202}$$

This formula is engraved on Boltzmann's tombstone at the Vienna Zentralfriedhof.

2

Variational Setting of Linear Steady-state Problems

2.1 Problem of the equilibrium of systems with a finite number of degrees of freedom

Recall all the necessary definitions of the classical mechanics of systems with a finite number of degrees of freedom, as they are presented, for example, in [GPS02]. Let us consider the simplest case of the equilibrium of a constrained mechanical system, consisting of n material points with coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$, with respect to (w.r.t.) some fixed Cartesian system with origin at the point O. The coordinates of the points are subject to s independent holonomic stationary constraints of the form

$$\Phi_i(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = 0, \quad i = 1, \ldots, s. \tag{2.1}$$

The configuration of the system can be characterized by k generalized coordinates $q_1, q_2, \ldots, q_k$, which are related to the Cartesian coordinates in the following way:

$$x_i = x_i(q_1, \ldots, q_k), \quad y_i = y_i(q_1, \ldots, q_k), \quad z_i = z_i(q_1, \ldots, q_k), \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Let F_i be the resultant of all the active forces acting on the material point with coordinates (x_i, y_i, z_i), and R_i denote the resultant force of the reactions of the constraints. In the equilibrium state we have

$$F_i + R_i = 0, \quad i = 1, 2, \ldots, n. \tag{2.3}$$

There are two essential problems in statics: to find the reaction of constraints in a given equilibrium state and to find the equilibrium state if the active forces applied to the system are known for all the configurations of the system. In this book, the last will be the more important problem.

Assume that the values F_i are either known or they are given as functions of the coordinates of points of the system

$$F_i = F_i(x_1, y_1, z_1, \ldots, x_n, y_n, z_n). \tag{2.4}$$

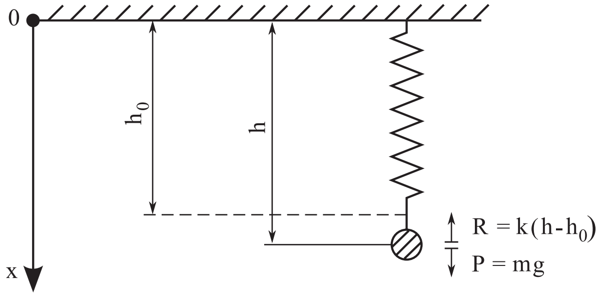

Fig. 2.1. A point mass on a spring

Suppose that the information about the reactions R_i is such that the set of relations (2.2)–(2.4), considered as a system of equations (after substituting (2.2) into (2.4) and (2.4) into (2.3) with respect to quantities $q_1, q_2, \ldots, q_k$ and unknown reactions (or part of them)), allows us to define one or several sets of generalized coordinates $q_1, q_2, \ldots q_k$, corresponding to the equilibria of the system.

Example 2.1. Consider the equilibrium of a point mass m, on a vertical spring in a constant gravitational field characterized by acceleration g (Figure 2.1).

Let the stiffness of the spring be equal to k, and the length be in the undeformed state h_0 and in an arbitrary deformed state h. Choose as a generalized coordinate $q = h - h_0$. The equilibrium equation is

$$mg - kq = 0. \tag{2.5}$$

This is a particular case of equation (2.3). It is derived with the additional hypothesis that the reaction force R is proportional to the relative elongation of the spring (the Hooke law):

$$R = k(h - h_0) = kq. \tag{2.6}$$

Suppose that the reaction of the spring is nonlinear, i.e., instead of (2.6) the following relation holds:

$$R = f(q), \tag{2.7}$$

where $f(q)$ is some function of elongation, the sign of which coincides with the sign of q for a physical reason. The absolute value of f increases together with the increase in the absolute value of q. Then, instead of the linear equation (2.5), we will have a nonlinear equation of the form

$$mg - f(q) = 0. \tag{2.8}$$

Example 2.2. Consider the equilibrium of a point mass in a constant gravity field, where the point slips without friction along a circle with the radius a in a vertical plane (Figure 2.2).

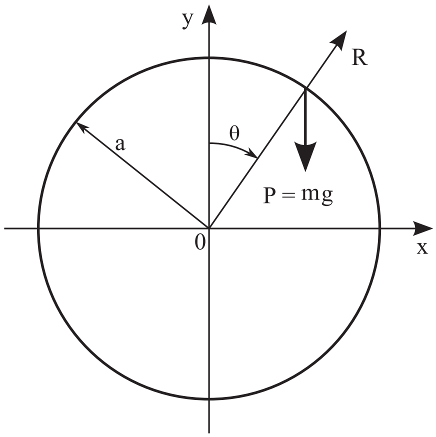

Fig. 2.2. A point mass on a ring

Let R be the absolute value of the reaction, and let $q = \theta$ be the generalized coordinate, defining the position of the point (Figure 2.2). Then, the system of equations defining the equilibrium state, projected onto the axes O_x, O_y, is

$$\begin{aligned} R\cos\theta - mg &= 0, \\ R\sin\theta &= 0. \end{aligned} \tag{2.9}$$

This system is nonlinear, and has two solutions. It becomes linear when $a = +\infty$, but in this case it has a continuum of solutions.

As is well known from analytical mechanics, one can deduce and solve equations on the Lagrange principle of virtual displacement instead of linear or nonlinear algebraic equations or systems of equations. In the current equilibrium problem, the coordinates of the points of the system are the numerical parameters, and the virtual displacements are their infinitesimal variations, satisfying the constraints. The relation between the infinitesimal changes in the Cartesian coordinates of the points of the system and the generalized coordinates follows from the formula (2.2) and the differentiation rule:

$$\delta x_i = \sum_{j=1}^{k} \frac{\partial x_i}{\partial q_j}\,\delta q_j, \quad \delta y_i = \sum_{j=1}^{k} \frac{\partial y_i}{\partial q_j}\,\delta q_j, \quad \delta z_i = \sum_{j=1}^{k} \frac{\partial z_i}{\partial q_j}\,\delta q_j, \quad i = 1, 2, \ldots, n. \tag{2.10}$$

It follows from the definition that the infinitesimal changes in the generalized coordinates (or, in short, the variations in the generalized coordinates), corresponding to the virtual displacements of the system, are independent.

Since the constraints imposed on the system are ideal, the work of the corresponding reactions R_j on the virtual displacements is equal to zero.

Hence, the virtual displacement equation, equivalent to the system (2.3), has the following form:

$$\sum_{i=1}^{n}(F_{ix}\delta x_i + F_{iy}\delta y_i + F_{iz}\delta z_i) = 0, \tag{2.11}$$

where (F_{ix}, F_{iy}, F_{iz}) are the components of the vector F_i. Putting (2.10) into (2.11), we obtain the following equation:

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\left(F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}\right)\delta q_j := \sum_{j=1}^{k} Q_j \delta q_j = 0, \tag{2.12}$$

where Q_j denotes the generalized force

$$Q_j = \sum_{i=1}^{n}\left(F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}\right). \tag{2.13}$$

The next step in the further analysis consists of a transition from the equation (2.12) to the problem of finding the stationary point of some function. Such a step is possible only when the forces F_i are conservative, i.e., when there exists a function $U(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ such that

$$F_{ix} = \frac{\partial U}{\partial x_i}, \quad F_{iy} = \frac{\partial U}{\partial y_i}, \quad F_{iz} = \frac{\partial U}{\partial z_i}. \tag{2.14}$$

It is well known that the function U exists if the following integrability conditions hold:

$$\frac{\partial F_{ix}}{\partial y_i} = \frac{\partial F_{iy}}{\partial x_i}, \quad \frac{\partial F_{iy}}{\partial z_i} = \frac{\partial F_{iz}}{\partial y_i}, \quad \frac{\partial F_{iz}}{\partial x_i} = \frac{\partial F_{ix}}{\partial z_i}. \tag{2.15}$$

The function U is called the *force potential*, and the function $\Pi = -U + C$ is the *potential energy* of the system. It is equal to the work done to transfer the system from some given state to the initial state. C is a constant, which can be taken to be zero. It is not difficult to find the expression for the function Π via the calculation of the work of the forces acting on the system:

$$\Pi(x_1, y_1, z_1, \ldots, x_n, y_n, z_n) = -\sum_{i=1}^{n}\int_{P_{0i}}^{P_i} F_i\, dr_i, \tag{2.16}$$

where dr_i denotes the vector with components (dx_i, dy_i, dz_i). The integral is a curvilinear one from the point P_{0i} with coordinates (x_{0i}, y_{0i}, z_{0i}) to the point P_i with coordinates (x_i, y_i, z_i). The choice of P_0 influences only the value of the constant C. The choice of the curve connecting point P_{0i} with point P_i is arbitrary, since the force is conservative.

Putting (2.14) into (2.11), one observes that in the equilibrium state the following equation holds:

$$\delta U = -\delta \Pi = 0, \tag{2.17}$$

where the symbol δ denotes the variation of the function, defined as the difference

$$U(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1, \dots, z_n + \delta z_n) - U(x_1, y_1, z_1, \dots, z_n)$$

linearized with respect to $\delta x_1, \dots$

From (2.17) we obtain the equations (2.11) if we take into account relations (2.14). Thus, the problem of finding the equilibrium state of a system is reduced to the problem of finding the stationary value of the force potential or the potential energy of the system. In order to find the stationary value of the function U (or Π), defined by (2.17), we calculate function Π for the examples given above.

In Example 2.1 for the linear spring we get

$$\Pi_1(q) = -U_1(q) = -mgq + \frac{kq^2}{2} + \text{const.} \tag{2.18}$$

For the nonlinear spring we have

$$\Pi_2 = -mgq + \int_{q_0}^{q} f(x)\,dx + \text{const.} \tag{2.19}$$

In Example 2.2 on the equilibrium of the point on the circle

$$\Pi_3 = mg\cos\theta + \text{const.} \tag{2.20}$$

We see that the function (2.18) has a single minimum at the point of equilibrium, as does the function (2.19). The function (2.20) has its maximum at the stationary point $\theta = 0$ and its minimum at the stationary point $\theta = \Pi$. These conclusions are immediate, but it is not difficult to demonstrate them using the condition of the strong minimum of the function at the point $q = q_0$:

$$\left.\frac{d\Pi}{dq}\right|_{q=q_0} = 0, \qquad \left.\frac{d^2\Pi}{dq^2}\right|_{q=q_0} > 0. \tag{2.21}$$

Clearly

$$\left.\frac{d^2\Pi_1}{dq^2}\right|_{q=q_0} = k > 0, \qquad \left.\frac{d^2\Pi_2}{dq^2}\right|_{q=q_0} = \frac{df}{dq} > 0, \tag{2.22}$$

$$\frac{d^2\Pi_3}{d\theta^2} = -mg\cos\theta \tag{2.23}$$

so that

$$\left.\frac{d^2\Pi_3}{d\theta^2}\right|_{\theta=0} = -mg < 0, \qquad \left.\frac{d^2\Pi_3}{d\theta^2}\right|_{\theta=\pi} = mg > 0. \tag{2.24}$$

Minimality conditions for the potential energy system derived at the stable point of equilibrium allow far deeper generalizations on the theory of systems with an infinite number of degrees of freedom, where they take the form of existence and uniqueness theorems for BVPs for systems of differential equations.

Now we begin the analysis of such systems.

2.2 Equilibrium of the simplest continuous systems governed by ordinary differential equations

2.2.1 Second-order problems

Systems with an infinite (nondenumerable) number of degrees of freedom will be called *continuous.* As the simplest example of such a system, consider a bar having the variable cross section $S = S(x)$. The bar is an axially symmetric body with the axis Ox (Figure 2.3). The bar is bounded by plane sections perpendicular to the axis Ox, the left end is fixed and the right is free. The material of the bar is linear elastic. Assume that the cross section $S(x)$ changes sufficiently smoothly so that displacements of all points at the section $x =$ const are equal and are defined by a function $u(x)$. Similarly, the normal stresses at the points at section x are assumed to be equal to $\sigma(x)$. The linearity of the mechanical properties of the material means that the linear relation

$$\sigma = E\varepsilon = E\frac{du}{dx}, \quad \varepsilon = \varepsilon(x) \tag{2.25}$$

is valid. In this equation $\varepsilon(x) = du/dx$ is the deformation (relative elongation of the infinitely small element in the direction of the axis Ox) and E is the Young modulus.

Suppose that the force per unit length is $F = F(x)$ parallel to the axis Ox.[1] Writing the equilibrium conditions for the infinitesimal element of bar with length dx, we deduce the following equilibrium equation:

$$\frac{d}{dx}(\sigma S) + F(x) = 0. \tag{2.26}$$

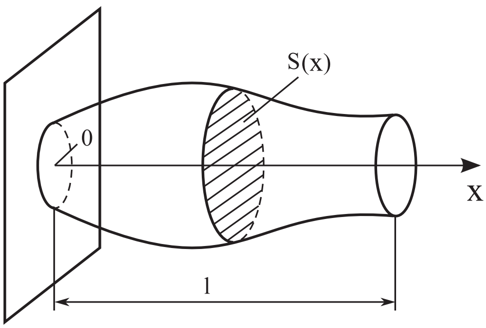

Fig. 2.3. Tension of a bar

[1] We follow the usual convention: the normal stress $\sigma(x)$ is the force per unit area applied by the material on the right side of section x on the material on the left.

Putting the relation (2.25) into the equation (2.26), we obtain an ordinary differential equation of the second order:

$$-\frac{d}{dx}\left[ES(x)\frac{du}{dx}\right] = F(x). \tag{2.27}$$

Suppose that the left end of the bar is fixed, then we will have the boundary condition

$$u(0) = 0. \tag{2.28}$$

The right end is free so that the second boundary condition has the form

$$\left.\frac{du}{dx}\right|_{x=l} = 0. \tag{2.29}$$

This condition follows from the relation (2.25) and the condition $S(l) > 0$.

The equation (2.27) is a generalization of equation (2.3) from Section 2.1. The main difference between those two equations is that the equation (2.3) holds at a finite number of points, and (2.27) holds for the continuum of points $x \in (0, l)$. The function $u(x)$ defined on the segment $(0, l)$ plays the part of the generalized coordinates. One of these coordinates is constrained by the condition (2.28).

Using the reasoning we used to obtain the equations (2.17), we can find the analogue for the problem (2.27)–(2.29). For this purpose, the variation $\delta u(x)$ of the function $u(x)$ is defined as the infinitesimal change in the function $u(x)$ at x:

$$\delta u(x) = v(x) - u(x) \equiv \varepsilon\eta(x), \quad \varepsilon \to 0. \tag{2.30}$$

The kinematic constraints on the variation $\delta u(x)$ are the continuity requirement and the boundary condition (2.28), i.e.,

$$\delta u(0) = 0. \tag{2.31}$$

Below, we require that the function $\delta u(x)$, i.e., $\eta(x)$, is differentiable. Notice that in some cases we will be forced to consider nondifferentiable function variations (e.g., "needle variations" [Fed78]).

Compose the analogue of the virtual works equation (2.11). Now the "forces" are the expressions on the left- and right-hand sides of the equation (2.27), and the points of the system run over the segment $[0, l]$. Therefore, instead of a sum, an integral appears over $[0, l]$:

$$-\int_0^l \frac{d}{dx}\left[ES(x)\frac{du}{dx}\right]\delta u(x)\,dx = \int_0^l F(x)\delta u(x)\,dx, \tag{2.32}$$

where $\delta u(x)$ is an arbitrary function in the space $\mathcal{D}((0, l))$ (see Definition 1.57), satisfying the stated limitations. Therefore, the equation (2.32) is completely equivalent to the equation (2.27). The last statement follows from the embedding theorem.

Applying integration by parts to the left-hand side of the equation (2.32), we obtain:

$$-\int_0^l \frac{d}{dx}\left[ES(x)\frac{du}{dx}\right]\delta u(x)\,dx = \int_0^l ES(x)\frac{du}{dx}\frac{d\delta u}{dx}\,dx - ES(x)\frac{du}{dx}\delta u(x)\Big|_0^l$$
$$= \int_0^l ES(x)\frac{du}{dx}\frac{d\delta u}{dx}\,dx. \qquad (2.33)$$

Notice that in the last transition we use the boundary conditions (2.28) and (2.29). If condition (2.29) is nonhomogeneous (this case corresponds to the loading by a force at the right end of the bar), i.e.,

$$\left(ES(x)\frac{du}{dx}\right)\Big|_{x=l} = P. \qquad (2.34)$$

Then, in the right-hand side of (2.33) the additional term $P\delta u(l)$ would appear.

Substitution of the expression (2.33) into the equation (2.32) gives

$$\int_0^l ES(x)\frac{du}{dx}\frac{d\delta u}{dx}\,dx = \int_0^l F(x)\delta u(x)\,dx \quad \forall \delta u(x). \qquad (2.35)$$

The equation (2.35) is equivalent to (2.32) only when its solution $u(x)$ is twice differentiable. The solution of equation (2.35) (not necessary twice differentiable) is called the *generalized solution* of the problem (2.25)–(2.27). The equation (2.35) is called the *integral identity* or *variational equation*, and corresponds to the boundary value problem (2.25)–(2.27).

The spaces V, in which the variational equations will be investigated, are the Sobolev spaces (see the definition (1.54)). We consider now the one-dimensional problems, and the corresponding functional spaces V being the subspaces of $H^1(0,l)$ for the equation (2.27).

It follows from the general definitions (1.53) and (1.54) of the norm in a Sobolev space that

$$\|f\|_{W^{m,p}(0,l)} \equiv \|f\|_{m,p,(0,l)} = \left(\sum_{k=0}^{m}\int_a^b \left|\frac{d^k f(x)}{dx^k}\right|^p dx\right)^{1/p}, \qquad (2.36)$$
$$\|f\|_{m,\infty,(0,l)} = \max_{k\le m}\left|\frac{d^k f}{dx^k}\right|, \quad |f| = \operatorname{ess\,sup}_x |f(x)|.$$

The equation (2.35) will be analyzed in the space $V \subset H^1(0,l)$ of the functions satisfying to the boundary condition (2.28). V is the Hilbert space with the inner product

$$(u,v)_{H^1} = \int_0^l uv\,dx + \int_0^l u'v'\,dx, \quad \|v\|_V^2 = (v,v)_{H^1}, \qquad (2.37)$$

where the prime denotes the first derivative). We shall also use the inner product in the space $L^2(0,l)$

$$(u,v)_{L^2} \equiv \langle u,v\rangle = \int_0^l uv\,dx. \tag{2.38}$$

Introduce now a linear functionals (linear forms) on the space V, denoted by

$$\langle f,v\rangle \equiv L(v). \tag{2.39}$$

Here, f is the element of the space V^* conjugate to the space V with respect to the given linear form $L(v)$. The functional $\langle f,v\rangle$, considered as a functional on the pair of spaces $\{V,V^*\}$, is bilinear, i.e., linear in f and in v. We shall also consider a bilinear functional of a general type on the space V, denoted by $a(u,v)$. The bilinear symmetric form on the space $V = H^1((0,l))$ is the functional on the left-hand side of the equation (2.35), i.e.,

$$a(u,v) = \int_0^l ES(x)u'v'\,dx. \tag{2.40}$$

The expression on the right-hand side of (2.35) is a linear functional on the space V $(F \in V^*)$

$$\langle F,v\rangle = \int_0^l F(x)v(x)\,dx. \tag{2.41}$$

Therefore, the variational equation (2.35) can be written in the following form:

$$a(u,\delta u) = \langle F,\delta u\rangle \quad \forall \delta u = v-u,\ v\in V,\ u\in V,\ F\in V^*. \tag{2.42}$$

We now perform the transition from variational equation (2.42) to the differential (2.35). Suppose that the solution (2.35) is twice differentiable (it has the generalized derivative, see Definition 1.57). Let the variation δu be in $\mathcal{D}(0,l)$. Performing the transformations (2.33) in the reverse order and taking into account the constraint $\delta u(x) = 0$ at point 0, we get the equation

$$\int_0^l \left[\frac{d}{dx}\left(ES(x)\frac{du}{dx}\right) - F\right]\delta u\,dx = 0 \quad \forall \delta u \in \mathcal{D}(0,l). \tag{2.43}$$

Using the closedness of V, we come to the equation (2.27).

Now, let $\delta u \in V$ in the equation (2.35). Using transformation (2.33) and taking into account the equation (2.27) and the boundary condition (2.28), we obtain the equation

$$\left.\left(ES(x)\frac{du}{dx}\delta u\right)\right|_{x=l} = 0 \quad \forall \delta u. \tag{2.44}$$

Since $\delta u(l)$ is an arbitrary variation, this equation immediately implies the condition (2.29) (clearly $E > 0$, $S(l) > 0$). Such boundary conditions, which are automatically fulfilled by the solutions to the variational equations, are called *natural* [CH53], contrary to the *forced* conditions of type (2.28), which must be added to the definition of the variational equation solution.

Now transform the equation (2.42) to an equation of the type (2.17), using the notation we introduced. For this purpose, consider the functional

$$J(v) = \frac{1}{2}a(v,\, v) = \frac{1}{2}\int_0^l ES(x)(v'(x))^2 dx \tag{2.45}$$

and define its variation δJ as the linear part of the increment of the given functional while going from element u to element $u + \delta u$, i.e.,

$$\delta J(u) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[J(u + \varepsilon \delta u) - J(u)\right]. \tag{2.46}$$

Using this definition, we can rewrite the equation (2.42) in the form

$$\delta\left[J(u) - \langle F, u\rangle\right] \equiv \delta \Pi(u) = 0, \quad u \in V, \tag{2.47}$$

where $\Pi(u)$ denotes the potential energy of the system:

$$\Pi(u) = J(u) - \langle F, u\rangle \equiv \frac{1}{2}a(u,u) - \langle F, u\rangle. \tag{2.48}$$

From this reasoning it is clear that the equations (2.35) and (2.47) are equivalent. The problem (2.47) is one of finding the stationary point of the functional Π. Imposing the physical constraints

$$E > 0, \quad S(x) \geq S_0 > 0, \quad S_0 = \text{const} \tag{2.49}$$

(the Young modulus and the area of the cross section do not vanish), one can prove, using well-known arguments from the elasticity theory [Tim87, Lov44], that the stationary point, defined by the equation (2.47), is the minimum and this minimum is unique. These statements and the theorem on the existence of the solution use the inequality

$$a(v,v) \geq \alpha |v|_V^2, \quad \alpha = \text{const} > 0, \tag{2.50}$$

following from the assumptions (2.49) and called the *positive definiteness property* of the bilinear functional (2.40) (see Chapter 1). Positive definiteness will be studied in Section 2.4 and the questions of the existence and uniqueness of the solution in Chapter 3.

2.2.2 Problems for fourth-order equations

As a more complicated example, consider the problem of the planar bending of the Euler–Bernoulli beam. Without going into detail on the derivation of the main relations (see, e.g., [Tim87]), we shall introduce only the equations and definitions necessary for our purpose.

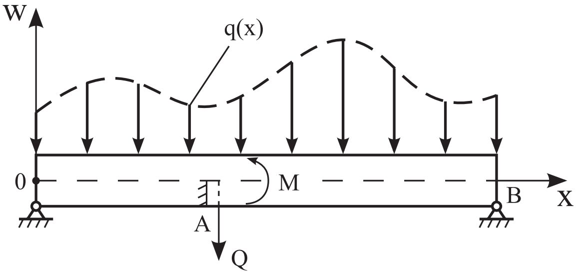

Fig. 2.4. Bending of a beam

The single function describing the bending of the beam under a load (Figure 2.4) is the transverse deflexion of the beam $w(x)$, being the vertical displacement of the points in the middle line of the beam. Internal force parameters, defining the action of the part of the beam AB onto part OA via the cross section at the point A, is the bending moment M and the shearing force Q. We have two equilibrium equations

$$\frac{dQ}{dx} = q, \quad \frac{dM}{dx} = Q, \tag{2.51}$$

which are obtained in the same way as the equation (2.26). The system of equations (2.51) is completed by the Bernoulli relation

$$M = EI \frac{w''}{[1 + (w')^2]^{3/2}}, \tag{2.52}$$

where I is the second moment of the cross section of the beam with respect to the axis, perpendicular to the plane of bending. In the geometrically linear deformation theory presented below, instead of the equation (2.52) we use the linearized relation

$$M = EIw''. \tag{2.53}$$

Assume (in general) that $I = I(x)$, and eliminate the moment M and shearing force Q from the relations (2.51) and (2.53). Then, we finally arrive at the fourth order equation

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 w}{dx^2}\right] = q(x) \tag{2.54}$$

called the *beam equation.*

Each end beam must have two boundary conditions. Typically, these are

- *Clamped* end (vertical displacement and rotation are zero)

$$w = 0, \quad w' = 0, \tag{2.55}$$

- *Pinned* or *simply supported* condition (the vertical displacement and the bending moment M are equal to zero)

$$w = 0, \quad w'' = 0, \tag{2.56}$$

- *Sliding* condition (the rotation and shearing force are zero)

$$w' = 0, \quad (Ew'')' = 0,$$

- The *free-end* condition (the shear force and the bending moment are equal to zero)

$$w'' = 0, \quad w''' = 0. \tag{2.57}$$

Perform the transition from the "local" problem (in the form of a differential equation in a domain with the given boundary conditions) to the variational setting for a particular case for the equation (2.54) with the boundary conditions

$$w|_{x=0} = 0, \quad \left.\frac{dw}{dx}\right|_{x=0} = 0, \tag{2.58}$$

$$\left.EI(x)\frac{d^2w}{dx^2}\right|_{x=l} = M_l, \quad \left.\frac{d}{dx}\left[EI(x)\frac{d^2w}{dx^2}\right]\right|_{x=l} = Q_l, \tag{2.59}$$

where M_l and Q_l are given constants.

Let the functional space, in which we will deduce and solve the variational equation, be $V \subset H^2(0,l)$, with the following constraints:

$$\text{if } v \in V \text{ then } v(0) = 0, \left.\frac{dv}{dx}\right|_{x=0} = 0. \tag{2.60}$$

Define the variation of the solution

$$\delta w = v - w = \varepsilon\eta, \quad \varepsilon \to 0,\ v \in V,\ w \in V. \tag{2.61}$$

The virtual work equation is obtained by multiplying (2.54) by the variation of solution $\delta w(x)$ and integrating over the segment $[0,l]$

$$\int_0^l (EI(x)w''(x))''\delta w(x)\,dx = \int_0^l q(x)\delta w(x)\,dx \quad \forall \delta w, \tag{2.62}$$

where, by definition, $w' \equiv \frac{dw}{dx}$.

Integration by parts twice, together with the boundary conditions (2.58) and (2.59), gives

$$\int_0^l EIw''\delta w''dx = \int_0^l q\delta w\,dx + M_l\delta w\prime(l) - Q_l\delta w(l). \tag{2.63}$$

This is an analogue of the equation (2.35) for the problem. The known linear functional becomes more complicated:

$$\langle F, u\rangle = \int_0^l qu\,dx + M_lu'(l) - Q_lu(l), \tag{2.64}$$

i.e., the element $F \in V^*$ cannot be identified explicitly. Now the bilinear functional becomes

$$a(u,v) = \int_0^l EI(x)u''(x)v''(x)\,dx. \tag{2.65}$$

With the notations (2.64) and (2.65) the problem still has the form (2.42).

The inverse transition from the equation (2.63) to the problem (2.54), (2.58) can be performed as in the transition from the equation (2.35) to the problem (2.27)–(2.29). Note that now the conditions (2.58) will be forced and the conditions (2.59) will be natural. Define the quadratic functional on V by the formula

$$J(v) = \frac{1}{2}a(v,\, v) = \frac{1}{2}\int_0^l EI(x)(v''(x))^2 dx. \tag{2.66}$$

Applying the definitions (2.46) and (2.64), we observe that the equation (2.63) can be written in the form

$$\delta[J(u) - \langle F, u\rangle] \equiv \delta\Pi(u) = 0, \quad u \in V, \tag{2.67}$$

i.e., the problem is again reduced to the problem of finding the stationary point of the functional Π, the potential energy of the system. Assuming that

$$E > 0, \quad I(x) \geq I_0 = \text{const} > 0, \tag{2.68}$$

we find that the stationary point of functional Π is unique and is the minimum (see Section 2.4).

At the end of this section, we would emphasize one special feature of these problems, i.e., that the natural conditions (involving force in the present problem) can be imposed at either end. From physical considerations, we know that the solution can exist only if the external loads are in equilibrium.

For the bar extension problem, it is sufficient to require the equilibrium of the longitudinal forces

$$\int_0^l F(x)\,dx + P(l) - P(0) = 0, \tag{2.69}$$

where

$$P(0) = ES(0)\frac{du}{dx}\bigg|_{x=0}, \quad P(l) = ES(l)\frac{du}{dx}\bigg|_{x=l}. \tag{2.70}$$

For the beam-bending problem, when external forces of the following type are given, at the ends, viz.

$$EI(0)w''|_{x=0} = M_0, \quad (EI(x)w''(x))'|_{x=0} = Q_0, \tag{2.71}$$

$$EI(l)w''|_{x=l} = M_l, \quad (EI(x)w''(x))'|_{x=l} = Q_l, \tag{2.72}$$

we have to impose the condition that the forces are in equilibrium

$$\int_0^l q(x)\,dx = Q_l - Q_0 \tag{2.73}$$

and that the moments are in equilibrium (e.g., with respect to the left end of the beam)

$$\int_0^l q(x)\,dx - Q_l - M_l + M_0 = 0. \tag{2.74}$$

The solution of this problem is not unique: we can add arbitrary vertical displacement and rotation as a rigid body movement. Formally, the solution is defined by two arbitrary constants. To define those two constants, we can prescribe the displacement and rotation at any point, or impose mean values for the whole beam, as follows:

$$\int_0^l w(x)\,dx = c_0, \qquad \int_0^l w'(x)\,dx = w(l) - w(0) = c_1, \tag{2.75}$$

where c_0 and c_1 are given constants.

If in the bar extension problem we only have to find the stress, then there is no need to use the Hooke law. We can solve the ordinary differential equation of the first order (2.26)

$$\frac{d}{dx}[\sigma(x)S(x)] + F(x) = 0 \tag{2.76}$$

with additional conditions (2.69) and (2.70), from which we obtain the initial condition for the Cauchy problem for the equation (2.76). For example,

$$\sigma(x)|_{x=0} = P(0) \tag{2.77}$$

(the stress at the point $x = l$ will be defined from the global equilibrium equation (2.69)). In the simplest case under consideration, the solution $\sigma(x)$ can easily be found:

$$\sigma(x) = \frac{1}{S(x)}\left[P(0) - \int_0^x F(s)ds\right]. \tag{2.78}$$

If, in the beam-bending problem, the force is given, and we again have to find only the force parameters (moment M and the shear force Q), then we can solve the system of ordinary differential equations (2.51) under additional conditions

$$M(0) = M_0, \quad Q(0) = Q_0, \quad M(l) = M_l, \quad Q(l) = Q_l \tag{2.79}$$

and with the global equilibrium equations (2.73) and (2.74). As in the previous case, from the relations (2.79), (2.73) and (2.74) we can obtain the Cauchy conditions for the system of the ordinary differential equations (2.51). But, contrary to the previous case, we have a new possibility: to build the boundary value problem

$$\frac{d^2M}{dx^2} = q(x), \quad 0 < x < l, \tag{2.80}$$

$$M(0) = M_0, \qquad M(l) = M_l, \tag{2.81}$$

for the bending moment M, excluding shear force Q from the equations (2.51).

Note that the functional spaces, in which we solve problems in the displacements by means of the prescribed forces at the ends, are the quotient space with the norm (1.82). In more earlier works such spaces are called *factor-spaces* with respect to the subspace of the rigid displacements (and, if necessary, rotations), e.g., [Mik64b].

Notice that, in principle, the problem can also be posed in the case where the conditions of the equilibrium of the system (of type (2.69) or (2.73) and (2.74)) are not fulfilled on the whole. In such cases, we have to use motion equations, i.e., to solve the problem for the acceleration and rotation of the bar on the whole, taking into account small deformations, instead of the equilibrium equations. Then contrary to the equations for the dynamics of the solids, we obtain partial differential equations in time and space.

2.3 3D and 2D problems on the equilibrium of linear elastic bodies

2.3.1 Strain

We shall analyze the stress–strain state of the body under a load occupying a domain Ω with boundary Σ. In the initial (unstressed) state, the body occupies the domain Ω_0 with boundary Σ_0. To simplify the further representation, we shall use, as a rule, the fixed Cartesian system of coordinates with the basis vector k_i, $i = 1, 2, 3$. The radius vectors of the particles in domain Ω_0 will be denoted by $a = a\{a^1, a^2, a^3\}$, a^i being the projections of the vector a onto the coordinate axes. The external forces acting on the deformed body imply the displacement of the particles of the body. As the result of these displacements, the domain Ω_0 is transformed into the domain Ω. The radius vectors of the particles into the domain Ω are denoted by $x = \{x^1, x^2, x^3\}$.

The main problem is to find the functions

$$x^i = x^i(a^1, a^2, a^3), \quad i = 1, 2, 3, \tag{2.82}$$

corresponding to the given external forces. If the motion of domain Ω is followed by changes in the corresponding distances between its particles, then it is said that the material in domain Ω is *deformable.* The local characteristics of the deformation of the domain Ω are the relative elongation ε of the infinitesimal segments dx near the point x:

$$\varepsilon = \frac{|dx| - |da|}{|da|}, \tag{2.83}$$

(where da is the element dx in the initial state), and the change in the angle γ between vectors da_1 and da_2, with origins at the same point a, which, under the load, goes to point x:

$$\gamma = \frac{dx_1 \cdot dx_2}{|dx_1||dx_2|} - \frac{da_1 \cdot da_2}{|da_1||da_2|}. \tag{2.84}$$

(dx_α is the element da_α in a current state. The dot means the inner product of the vectors.)

To find the main characteristics ε and γ, calculate the difference β of squares of the lengths before and after deformation (without load and under load),

$$\beta = |dx|^2 - |da|^2 \equiv ds^2 - da^2 \tag{2.85}$$

for an arbitrary infinitesimal element dx, going from point x. Since $x = x(a)$, then

$$dx = \frac{\partial x}{\partial a^i} da^i. \tag{2.86}$$

We shall use the Einstein summation convention: a repeated Latin index implies summation over 1, 2, 3, a repeated Greek index implies summation over 1, 2. If for some reason this convention is not followed we underline the indices.

From the formula (2.86), we have

$$ds^2 = |dx|^2 = dx \cdot dx = \frac{\partial x}{\partial a^i} \cdot \frac{\partial x}{\partial a^j} da^i da^j = \frac{\partial x^q}{\partial a^i} \frac{\partial x^q}{\partial a^j} da^i da^j. \tag{2.87}$$

Moreover,

$$ds_0^2 = da \cdot da = da^p da^p. \tag{2.88}$$

Substitution of the formulae (2.87) and (2.88) into (2.85) gives

$$\beta = \frac{\partial x^q}{\partial a^i} \frac{\partial x^q}{\partial a^j} da^i da^j - da^p da^p = \left(\frac{\partial x^q}{\partial a^i} \frac{\partial x^q}{\partial a^j} - \delta_{ij} \right) da^i da^j, \tag{2.89}$$

where δ_{ij} is the Kronecker symbol: $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. The set of values

$$\frac{1}{2} \left(\frac{\partial x^q}{\partial a^i} \frac{\partial x^q}{\partial a^j} - \delta_{ij} \right) \equiv \varepsilon_{ij}^G \tag{2.90}$$

represents the set of components of a tensor of the second order, called the *Green tensor of finite deformation.*

Now, we express the values ε_{ij}^G in terms of the components of the vector of displacement $u = x(a) - a = u^i k_i$:

$$\frac{\partial x^i}{\partial a^j} = \frac{\partial u^i}{\partial a^j} - \delta_{ij}. \tag{2.91}$$

Substitution of the formula (2.91) into (2.90) gives

$$\varepsilon_{ij}^G = \frac{1}{2} \left(\frac{\partial u^i}{\partial a^j} + \frac{\partial u^j}{\partial a^i} + \frac{\partial u^q}{\partial a^i} \frac{\partial u^q}{\partial a^j} \right). \tag{2.92}$$

Define now the unit vector $\nu_0 = \nu_0^i k_i$, oriented along the element da, i.e.,

$$da = \nu_0 ds_0, \quad da^i = \nu_0^i ds_0. \tag{2.93}$$

Then

$$\beta = 2\varepsilon_{ij}^{G}\nu_0^i\nu_0^j ds_0^2. \tag{2.94}$$

Let

$$da_1 = \nu_{01}ds_{01}, \quad da_2 = \nu_{02}ds_{02}. \tag{2.95}$$

Then

$$\gamma = \nu_{01}^i\nu_{02}^j\left[\left(2\varepsilon_{ij}^G + \delta_{ij}\right)\frac{ds_{01}}{ds_1}\frac{ds_{02}}{ds_2} - \delta_{ij}\right]. \tag{2.96}$$

Thus, knowing the components ε_{ij}^G of the Green tensor of deformations, in view of the formulae (2.94) and (2.96), we can define the local characteristics of deformation γ and β and the relative elongation:

$$\varepsilon = \sqrt{1 + 2\varepsilon_{ij}^G\nu_{0i}\nu_{0j}} - 1. \tag{2.97}$$

In this paragraph, we stay within the framework of the linear theory: using the assumption that the components u^i of the vector of displacements u and the first derivatives of u^i are infinitesimal, we linearize all the equations and conditions, neglecting the products of such functions and their derivatives. Linearization of the expression on the right-hand side of the equation (2.92) gives the tensor of small deformations with components

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial a^j} + \frac{\partial u^j}{\partial a^i}\right), \tag{2.98}$$

which is called the *Cauchy tensor of deformations* and will be widely used later.

Recall the *Cesaro formula*, which is very useful for a number of problems. This formula relates the components u^i of the small displacements vector and the tensor of small deformations ε_{ij}

$$u^i(a_1) = u^i(a_0) + \omega_{ik}^0(a_1^k - a_0^k) + \int_{\overset{\frown}{M_0M_1}}\left\{\varepsilon_{ir} + (a_1^k - a^k)\left[\frac{\partial\varepsilon_{ir}}{\partial a^k} - \frac{\partial\varepsilon_{kr}}{\partial a^i}\right]\right\}da^r, \tag{2.99}$$

where a_0 is a fixed point in the domain Ω_0, in which the vector of displacements $u^i(a_0)$ and the components of the tensor of small rotations are assumed to be known. By definition,

$$\omega_{ij}(a_0) \equiv \omega_{ij}^0 = (\partial u^i/\partial a_j - \partial u^j/\partial a_j)/2.$$

The integral in (2.99) is taken over the curve $\overset{\frown}{M_0M_1}$, connecting the points M_0 and M_1 with the radius vectors a_0 and a_1. The conditions of compatibility which results from this formula (independence of the integral (2.99) on the curve, connecting point M_0 with point M_1), called the Saint-Venant compatibility conditions, have the form

$$\varepsilon_{ir,ks} - \varepsilon_{kr,is} = \varepsilon_{is,kr} - \varepsilon_{ks,ir}. \tag{2.100}$$

(The comma means differentiation with respect to the variable a with the index coming after the comma. If there are two indices after the comma, then it means a derivative of the second order: e.g., $f_{,ks} = \partial^2 f/\partial a^k \partial a^s$.) The details of the Cesaro formula, the compatibility conditions, and another useful formula derivation are given, for example, in [Tim87].

2.3.2 Stresses

To describe the internal forces (stresses), related to the deformations of a solid, one usually uses the method of sections: the domain Ω is dissected by an arbitrary smooth surface $\tilde{\Sigma}$ into two subdomains Ω_1 and Ω_2. The following assumptions are used:

1. The effect of Ω_1 onto Ω_2 is limited to the surface $\tilde{\Sigma}$.
2. The effect of Ω_1 onto Ω_2 is represented by surface forces (not by moments), distributed over the surface $\tilde{\Sigma}$ with density t.
3. The vector t depends only on the point M of the surface $\tilde{\Sigma}$ and on the unit vector ν, which is normal to $\tilde{\Sigma}$ at point M (ν is directed into Ω_2), but does not depend on the local structure of the surface $\tilde{\Sigma}$. In another words, if we dissect the domain Ω by an another surface $\tilde{\Sigma}'$, which is different from $\tilde{\Sigma}$, but shares a common point M and a common tangent plane at this point, then at M we obtain the same density of surface forces $t(M, \nu)$.

Let us consider the equilibrium of an infinitely small tetrahedron cutout of the domain in the neighborhood of the point M by three plane sides parallel to the coordinate axes and the fourth plane side with the normal ν. Using a well-known procedure [Tim87], we find that dependence t on vector ν is linear:

$$t(M, \nu) \equiv t^{(\nu)} = t_i \nu_i, \tag{2.101}$$

where $t^{(\nu)}$ is the vector of the density of the surface forces on the side with normal ν, and t_i is the density force on the side perpendicular to the ith coordinate axis ($i = 1, 2, 3$). The matrix of the numbers t_{ij}, representing the set of projections of the vectors $t_i = t_{ij} k_j$ onto the coordinate axes, forms the set of components of the second-order tensor called the *Euler tensor of the internal stresses* $\hat{t}$ (or simply the *stress tensor*). Below we will also use the denotation $\sigma_{ij} = t_{ij}$. In the moving body choose an arbitrary subdomain Ω_t with a boundary Σ_t, and apply Newton's second law of motion. Let ρ denote the density of material, $\dot{u} = \partial u/\partial t$ denote the velocity of the particle, and $\ddot{u} = \partial^2 u/\partial t^2 = \partial \dot{u}/\partial t$ be its acceleration. Then

$$\int_{\Omega_t} \rho \ddot{u} \, d\Omega = \int_{\Omega_t} \rho F \, d\Omega + \int_{\Sigma_t} t^{(\nu)} \, d\Sigma, \tag{2.102}$$

where $t^{(\nu)}$ is the surface density of force on the boundary Σ_t with outward normal ν, F is the density of the body forces, assumed known (in the general case these are functions of the space coordinates).

Applying the Ostrogradski–Gauss theorem to transform the surface integral in (2.102) into a volume integral and noting that the domain Ω_t is arbitrary, we obtain the differential equation of motion

$$\operatorname{div} \hat{t} + \rho F = \rho \ddot{u} \tag{2.103}$$

or, in a component form,

$$\frac{\partial \sigma_{ij}}{\partial x^i} + \rho F_j = \rho \frac{\partial^2 u_j}{\partial t^2}. \tag{2.104}$$

When the inertial forces are zero, we have the equation of equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x^i} + \rho F_j = 0. \tag{2.105}$$

Applying the equation (2.102) to the infinitely thin layer near the surface Σ of the body Ω, where the density P of the prescribed surface forces (e.g., the pressure) is known, we obtain the boundary condition for the stresses:

$$t^{(\nu)} \cdot \nu|_{\Sigma} = P \tag{2.106}$$

or, in a component form,

$$\sigma_{ij} \nu_j|_{\Sigma} = P_i. \tag{2.107}$$

From the equation

$$\int_{\Omega_t} x \times \rho \ddot{u} \, d\Omega = \int_{\Omega_t} x \times \rho F \, d\Omega + \int_{\Sigma_t} x \times t^{(\nu)} \, d\Sigma, \tag{2.108}$$

(where $\times$ denotes the vector cross product) following from the rate of change of moment of momentum, we obtain (see, e.g., [Ger73]) the symmetry of the stress tensor:

$$\sigma_{ij} = \sigma_{ji}.$$

Consider now the state of the system, close to the unstressed state. Suppose that the displacements and their first- and-second order derivatives are small. Using the linearization method, we obtain the so-called *geometrically linear theory.* This theory operates with the Cauchy tensor of deformations (2.98), surface forces calculated per unit area of the nondeformed body, and body forces and mass density calculated per unit volume of the nondeformed body. Motion and equilibrium equations and boundary conditions are referred to the nondeformed body too, i.e., independent spacial variables are $a \in \Omega_0$ (see, e.g., [Ger73]).

In this section, we consider only the geometrically linear theory.

2.3.3 Strain–stress relation (the generalized Hooke law)

Define an *elastic material* as a continuum, in which

- Stresses in the neighborhood of some point depend only on the deformation of this neighborhood
$$\sigma_{ij} = F_{ij}(\{\varepsilon_{kl}\}) \tag{2.109}$$
- Stresses are independent of the path from the initial state to the deformed state

Linearizing the governing equation (2.109) near the initial state w.r.t. the deformations ε_{kl}, we obtain the formula

$$\sigma_{ij} = \left.\frac{\partial F_{ij}}{\partial \varepsilon_{kl}}\right|_{\varepsilon_{kl}=0} \cdot \varepsilon_{kl} \equiv a_{ijkl}\varepsilon_{kl}. \tag{2.110}$$

In the linearization process we use the assumption that in the initial state the stresses and strains are zero. The coefficients a_{ijkl} in the formula (2.110) are called the *moduli of elasticity.* They have the symmetry

$$a_{ijkl} = a_{ijlk} = a_{jikl} \tag{2.111}$$

derived from the symmetry of the tensors σ_{ij} and ε_{kl}. The second property, the definition (independence of path), gives the additional symmetry

$$a_{ijkl} = a_{klij}. \tag{2.112}$$

The governing equation (2.110) is also called the *generalized Hooke law.* Elastic materials without dissipation and governed by the law (2.110) are called *linear elastic* materials.

2.3.4 Formulation of boundary value problems

These equations and relations allow us to formulate the main BVPs of the linear theory of elasticity.

In the general case, we have 15 unknown functions, σ_{ij}, ε_{ij}, u_i, and 15 equations to find them:

$$\frac{\partial \sigma_{ij}}{\partial a^i} + \rho F_j = \rho \ddot{u}_j, \tag{2.113}$$

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}, \tag{2.114}$$

$$\varepsilon_{kl} = (u_{i,j} + u_{j,i})/2. \tag{2.115}$$

The equation (2.113) must be complemented by the boundary conditions. In the simplest case, these conditions are the relations between the external prescribed forces and the internal stresses (we will omit the index t in the notation of the domain and its boundary in the geometrically linear theory):

$$\hat{\sigma} \cdot \nu|_{\Sigma_\sigma} = P. \tag{2.116}$$

For the part Σ_u the displacements are prescribed

$$u|_{\Sigma_u} = g. \tag{2.117}$$

The functions P and g depend on the coordinates of the surface points.

Note that in practice we sometimes have the boundary conditions in which a part of the components of the surface forces and a part of the components of the displacements are prescribed, but the number of boundary conditions in space problems is equal to three. In "contact problems" the boundary conditions take the form of inequalities. This topic will be considered in Chapter 4. In this section, we consider only the problems for equations (2.113)–(2.115) with the boundary conditions (2.116) and (2.117).

In the dynamic problems, when we cannot omit the inertia forces, the problem has to be completed by the initial conditions

$$u|_{t=t_0} = u_0(a), \quad \left.\frac{\partial u}{\partial t}\right|_{t=t_0} = u_1(a), \quad a \in \Omega \tag{2.118}$$

or by some periodicity conditions if we are investigating steady-state vibration processes.

Thus, the main variant of the linear dynamic problem in the theory of elasticity consists of solving the equations (2.113)–(2.115) under the boundary conditions (2.116) and (2.117) and the initial conditions (2.118). The essential static problem consists of solving the equations (2.113)–(2.115) with the zero right-hand part of equation (2.113) under the boundary conditions (2.116) and (2.117).

Assume that

$$\Sigma_\sigma \cup \Sigma_u = \Sigma \equiv \partial\Omega, \tag{2.119}$$

where $\partial\Omega$ is the boundary of the domain Ω. There is a special case in which

$$\Sigma_\sigma = \Sigma.$$

We already met such a situation at the end of Section 2.2. For the well-posedness of the problem, we must add to all these equations and conditions the following equilibrium equations for the given external forces and their moments:

$$\int_\Omega \rho F \, d\Omega + \int_\Sigma P \, d\Sigma = 0, \tag{2.120}$$

$$\int_\Omega a \times \rho F \, d\Omega + \int_\Sigma a \times P d\Sigma = 0. \tag{2.121}$$

For the uniqueness of the displacement field, it is necessary to exclude a rigid body translation and rotation. It is necessary either to prescribe the displacements and the rotation in the neighborhood of some point or require the means of these quantities over the domain to be zero:

$$\int_\Omega u \, d\Omega = 0, \quad \int_\Omega a \times u \, d\Omega = 0. \tag{2.122}$$

In practice, the system of equations (2.113)–(2.115) is rarely used. More often one uses formulations where we exclude from the equations either σ_{ij} and ε_{ij}. If, for example, we express the stresses in terms of strains, and the strains in terms of displacements, we obtain the following system for displacements (for static problems):

$$-\frac{1}{2}\frac{\partial}{\partial a^j}\left[a_{ijkl}\left(\frac{\partial u_k}{\partial a^l}+\frac{\partial u_l}{\partial a^k}\right)\right]=\rho F_i \tag{2.123}$$

with the boundary conditions

$$\frac{1}{2}[a_{ijkl}(u_{k,l}+u_{l,k})]\nu_j|_{\Sigma_\sigma}\equiv\sigma_{ij}(u)\nu_j|_{\Sigma_\sigma}=P_i, \tag{2.124}$$

$$u|_{\Sigma_u}=g. \tag{2.125}$$

If the deformations ε_{ij} and displacements u are expressed in terms of stresses, we arrive at a problem for the stresses. This procedure is often used in 2D problems.

For space problems, the system of equations is complicated, and only in the last few years have been proposed some ways of analyzing and solving such problems [Pob84].

For an isotropic body [Lov44], the stress–strain coefficients are

$$a_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{2.126}$$

where λ and μ are the Lamé constants and δ_{ij} is the Kronecker symbol. The stresses are obtained by substituting (2.126) into the generalized Hooke law (2.114):

$$\sigma_{ij}=\lambda\Theta\delta_{ij}+2\mu\varepsilon_{ij},\quad \Theta=\operatorname{div}u=\varepsilon_{kk}. \tag{2.127}$$

Solving these equations with respect to strains, we find

$$\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}(\sigma_{kk})\delta_{ij}, \tag{2.128}$$

where

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \tag{2.129}$$

is the Young modulus and

$$\nu=\frac{\lambda}{2(\lambda+\mu)} \tag{2.130}$$

is the Poisson ratio. The system of equations for the stresses consists of the set of equilibrium equations

$$\frac{\partial\sigma_{ij}}{\partial a^i}+\rho F_j=0, \tag{2.131}$$

and the equation obtained by substitution of the expressions (2.128) into the Saint-Venant compatibility equation (2.100)

$$\triangle\sigma_{ij} + \frac{1}{1+\nu}(\sigma_{kk,ij}) + \frac{\nu}{1-\nu}(\operatorname{div}\rho F)\sigma_{ij} + \rho(F_{i,j} + F_{j,i}) = 0. \qquad (2.132)$$

The system is completed by the force boundary conditions

$$\sigma_{ij} \cdot \nu_j|_{\Sigma_\sigma} = P_i. \qquad (2.133)$$

The equation (2.132) is called the *Beltrami–Mitchell equation.*

So, we have a problem: to find the six components of the stress tensor σ_{ij}, we have nine equations (2.131) and (2.132). In theoretical research it is possible to use this system. To solve technical problems, it is necessary to form independent combinations of the unknown functions and to write the system of three independent compatibility conditions (as done by Beltrami [Lov44]) or to use another procedure to eliminate the unknown functions ε_{ij} and u_i in order to obtain six equations for the six unknown functions σ_{ij} [Pob84].

2.3.5 2D problems of the linear theory of elasticity

In 2D problems there are only two independent variables, and the number of unknown functions is also decreased. We will consider plane strain deformation, generalized plane stress, the torsion problem, and the bending of a thin plate.

Plane strain deformation of an elastic body

Plane strain deformation is characterized by the following field of displacements:

$$u_1 = u_1(x, y), \quad u_2 = u_2(x, y), \quad u_3 = 0. \qquad (2.134)$$

This state of plane deformation arises in the middle part of a long cylindrical body under external forces which do not depend on the axial x_3- or z-coordinate. The axes Ox and Oy of the Cartesian system are placed in the cross section, perpendicular to the axis Oz. From the expression (2.134), we obtain

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0, \quad \Theta = \operatorname{div} u = \varepsilon_{\alpha\alpha}. \qquad (2.135)$$

(Recall that by our convention, Greek indices range over 1, 2.) The generalized Hooke law is

$$\sigma_{\alpha\beta} = \lambda\Theta\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta}. \qquad (2.136)$$

The component σ_{33} of the strain tensor is defined by the components $\sigma_{\alpha\beta}$ according to the formula

$$\sigma_{33} = \lambda\Theta = \frac{\lambda}{2(\lambda+\mu)}\sigma_{\alpha\alpha}, \qquad (2.137)$$

which follows from the condition $\varepsilon_{33} = 0$.

The equilibrium equations and the boundary conditions preserve their forms, if the Latin indices are replaced by the Greek ones:

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho F_\alpha = 0, \qquad (x_1,\, x_2) \in \Lambda, \tag{2.138}$$

$$\sigma_{\alpha\beta} \cdot \nu_\beta|_{\Gamma_\sigma} = P_\alpha(x), \quad x \in \Gamma_\sigma, \tag{2.139}$$

$$u_\alpha|_{\Gamma_u} = g_\alpha(x), \quad (x_1,\, x_2) \in \Gamma_u, \tag{2.140}$$

where Λ is now a 2D (open set) domain forming the cross section of the cylinder, its boundary Γ being a flat curve.

State of generalized plane stresses

Consider a thin plate with the external force actions distributed symmetrically with respect to the middle plane of the plate. Choose the coordinate system with its origin at a point in the middle plane and with the axes Ox_1 and Ox_2 in the middle plane, the axis Ox_3 being perpendicular to this plane. Then, the domain Ω occupied by the plate is defined by

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \Lambda;\; -h \le x_3 \le h\}. \tag{2.141}$$

It is assumed that the edges $x_3 = \pm h$ of the plate are free of external forces.

By supposition, the thickness h is small with respect to the dimensions of the domain Λ. If the external forces are sufficiently smooth then the variation of the displacements, strains, and stresses along Ox_3 is small relative to the changes in the plane Ox_1x_2. Therefore, we can investigate the means of these quantities over the thickness

$$u^*_\alpha = \frac{1}{2h} \int_{-h}^{h} u_\alpha(x_1, x_2, x_3)\, dx_3, \tag{2.142}$$

$$\sigma^*_{\alpha\beta} = \frac{1}{2h} \int_{-h}^{h} \sigma_{\alpha\beta}(x_1, x_2, x_3)\, dx_3. \tag{2.143}$$

Using these in the governing equations, we again obtain a problem of the type (2.138)–(2.140). In the Hooke law (2.136) the combination

$$\tilde{\lambda} = 2\lambda\mu/(\lambda + 2\mu) \tag{2.144}$$

appears, and instead of the Lamé constant λ and instead of the equation $\varepsilon_{33} = 0$ we have

$$\sigma_{33} = 0. \tag{2.145}$$

This equation is valid up to $O(h^2)$. The proofs of these statements can be found, for example, in [Lov44].

Notice that, introducing the Airy function U for the stresses by the formula

$$\sigma_{11} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial x \partial y} \tag{2.146}$$

(where $x_1 \equiv x$, $x_2 \equiv y$) and assuming that there are no body forces, we exactly satisfy the equilibrium equations (2.138). From the Beltrami–Mitchell conditions we obtain the biharmonic equation for the function $U(x, y)$

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} \equiv \Delta\Delta U = 0, \quad (x, y) \in \Lambda. \tag{2.147}$$

From conditions (2.139) we obtain Dirichlet-type boundary conditions for the equation (2.147)

$$U|_\Gamma = g_1(x, y), \qquad \left.\frac{\partial U}{\partial \nu}\right|_\Gamma = g_2(x, y), \quad (x, y) \in \Gamma, \tag{2.148}$$

where the functions g_1 and g_2 are defined by the prescribed external forces on the curve Γ:

$$g_1(s) = c + ax(s) + by(s) + \int_{M_0}^{M} \left[-\frac{\partial x}{\partial s} \int_{M_0}^{\tilde{M}} P_2(\tilde{s})\, d\tilde{s} + \frac{\partial y}{\partial s} \int_{M_0}^{\tilde{M}} P_1(\tilde{s})\, d\tilde{s} \right] ds, \tag{2.149}$$

$$g_2(s) = \left[a - \int_{M_0}^{M} P_2(s) ds \right] \frac{\partial x}{\partial \nu} + \left[b + \int_{M_0}^{M} P_1(s) ds \right] \frac{\partial y}{\partial \nu}, \tag{2.150}$$

where a, b, and c are arbitrary constants. At every point of the curve we introduce local Cartesian system with the basis vectors s being tangent to the curve Γ and ν being normal to it. The curve Γ is defined by the parametric equations

$$x = x(s), \quad y = y(s), \tag{2.151}$$

where s is the distance of the current point from a fixed point M_0 on the curve. External actions are also defined as functions of the parameter s. The integrals in the expressions (2.149) and (2.150) are curvilinear ones from the point M_0 on the curve Γ to the current point M or $\tilde{M}$ on the same curve.

Saint-Venant torsion bar problem

Let the domain Ω be a right cylinder with an arbitrary cross section. The lateral boundary of the cylinder is denoted by Σ_2. Let Σ_0 and Σ_1 be the end-wall sections. Assume that the surface forces on Σ_0 and Σ_1 are prescribed and are statically equivalent to a moment parallel to the axis of the cylinder, and the lateral boundary Σ_2 is free.

The boundary value problem consists of the solution of the 3D equations of elasticity with the boundary conditions described above.

To solve this problem, Saint-Venant introduced simplifying hypotheses. To formulate these hypotheses, take the axis Oz along the cylinder, and Ox and Oy in the plane of the cross section.

The first Saint-Venant hypothesis consist of the supposition that every cross section of the cylinder by the plane $z \equiv x_3 = \text{const}$ remains plane and the right angles to the axis and is twisted through some angle α proportional to z. We find the following displacement field:

$$u_1 = -\alpha yz, \quad u_2 = \alpha xz, \quad \alpha = \text{const}, \tag{2.152}$$

$$u_3 = \alpha\phi(x, y). \tag{2.153}$$

The formula (2.153) represents the fact that all the points on the straight line $x = \text{const}$, $y = \text{const}$, move along the axis of the cylinder by the same amount. The function $\phi(x, y)$ is called the *warping function* or simply the *warping.*

We calculate the strains from the given field of displacements, and the stresses from the strains with the Hooke law. Using the equilibrium equation and the boundary conditions on Σ_α, we find that the function $\phi(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \equiv \Delta\phi(x, y) = 0 \tag{2.154}$$

and the boundary condition

$$\left.\frac{\partial\phi}{\partial\nu}\right|_\Gamma = (y\nu_1 - x\nu_2)|_\Gamma, \tag{2.155}$$

where Γ is the boundary of the domain Λ in the plane Oxy occupied by the cross section of the bar, and $\nu = (\nu_1, \nu_2)$ is the outward unit vector normal to the boundary Γ directed out of the domain Λ.

Introduce now two new functions: the function $\Theta(x, y)$ related to the function $\phi(x, y)$ by the formula

$$-y + \frac{\partial\phi}{\partial x} = \frac{\partial\Theta}{\partial y}, \quad x + \frac{\partial\phi}{\partial y} = -\frac{\partial\Theta}{\partial x} \tag{2.156}$$

and the function $\psi(x, y)$, conjugate to the function $\phi(x, y)$, satisfying the Cauchy–Riemann conditions

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}, \quad \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}. \tag{2.157}$$

Using the equation (2.154), we find that the function Θ satisfies the Poisson equation

$$\Delta\Theta(x, y) = -2, \quad (x, y) \in \Lambda \tag{2.158}$$

and, for a simply connected domain Λ, the absence of external forces on the lateral surface of the bar means that

$$\Theta|_\Gamma = \text{const.} \tag{2.159}$$

The nonzero components of the strain tensor are

$$\sigma_{13} = \mu\alpha \frac{\partial \Theta}{\partial y}, \quad \sigma_{23} = -\mu\alpha \frac{\partial \Theta}{\partial x}. \tag{2.160}$$

This shows that the choice of the constant in the condition (2.159) does not affect the strain. We take it to be zero. The function ψ, conjugate to the function ϕ, also satisfies the Laplace equation. The boundary condition for it follows from the formula

$$\psi(x, y) = \Theta(x, y) + (x^2 + y^2)/2 \tag{2.161}$$

and can be transformed to

$$\psi|_\Gamma = (x^2 + y^2)/2|_\Gamma. \tag{2.162}$$

To find the constant α presented in the torsion problem, Saint-Venant proposed the following principle: *If an elastic body is in equilibrium under two different but statically equivalent external forces, distributed in a small domain, then the stresses and strains are the same at points far from the loaded domain.*

Note that there are qualitative estimates of the "smallness" of the loaded domain, and of the difference between stresses and strains for two different statical equivalent loads, depending on the distance from the loaded domain, etc.

It follows from the Saint-Venant principle that we can satisfy the boundary condition using the integral equilibrium condition

$$M = \int_\Lambda\!\!\int (x\sigma_{23} - y\sigma_{13})\, dx\, dy = 2\mu\alpha \int_\Lambda\!\!\int \Theta(x, y)\, dx\, dy, \tag{2.163}$$

where M is the prescribed twisting moment. This equation means, first that the external loads on the end sections Σ_0 and Σ_1 are statically equivalent to the moments M and $-M$ parallel to the cylinder axis Oz. Secondly, the solution of the bar torsion problem for the boundary condition (2.163) corresponds to an infinity of systems of loads statically equivalent to the moments M and $-M$.

Sophie Germain theory of thin plate and thin membrane bending

Consider again the domain Ω, described as in the precedent problem (state of generalized plane stress) with the same choice of coordinate system, i.e., with the origin on the middle plane, taken as the plane $Ox_1x_2 \equiv Oxy$, and the axis $Ox_3 \equiv Oz$ perpendicular to the middle plane. Formally, the domain Ω is defined by the relation (2.141), but now, contrary to the case of generalized plane stress, the plate is assumed to be loaded on the boundaries $z = \pm h$.

Let the thickness $2h$ be small with respect to the maximum dimensions l of the plane in its plane. Introduce the resultant of the internal forces over the thickness

$$R_i = \int_{-h}^{h} \sigma_{i\alpha}\nu_\alpha dz = Q_{i\alpha}\nu_\alpha \tag{2.164}$$

and the moments of these forces

$$M_i = \int_{-h}^{h} E_{i3k}\sigma_{k\alpha}\nu_\alpha dz = M_{i\alpha}\nu_\alpha, \tag{2.165}$$

where $\nu = (\nu_1, \nu_2, 0)$ is normal to the intersection of the plate by the cylindrical surface, with the lateral boundary elements parallel to the axis Oz, and ε_{ijk} are the Levi-Civita symbols

$$\varepsilon_{ijk} = \begin{cases} -1, & \text{if } i,\ j,\ k \text{ is even,} \\ +1, & \text{if } i,\ j,\ k \text{ is odd,} \\ 0, & \text{if at least two for } i,\ j,\ k \text{ coincide.} \end{cases}$$

A sequence i, j, k is said to be even (odd) if it can be obtained from 1, 2, 3 by an even (odd) number of permutation.

If we write the equilibrium equations for an arbitrary subdomain of the plate, dissected by the cylindric surface parallel to the axis Oz and by the equations of the resultant vector and resultant moment of the forces applied to this subdomain, we obtain the system of differential equations

$$Q_{i\alpha,\alpha} + q_i = 0, \tag{2.166}$$

$$\varepsilon_{ik\alpha}Q_{k\alpha} + M_{i\alpha,\alpha} + m_i = 0, \tag{2.167}$$

where

$$q_i = \int_{-h}^{h} F_i(x,y,z)dz + q_i^+ + q_i^- \tag{2.168}$$

is the resultant of the forces applied to the segment $-h \le z \le h$, $x = \text{const}$, $y = \text{const}$ of the bar, q_i^+ is the density of the surface forces, prescribed at the upper end, q_i^- is the density at the lower end $z = -h$, F_i is the density of the body forces, and

$$m_i = \int_{-h}^{h} \varepsilon_{i3k} z F_k(x,y,z)dz + m_i^+ + m_i^- \tag{2.169}$$

is the sum of all the moments applied to the same segment. The first two equations of the system (2.166) are an already known system of equations for the plane problem (up to the multiplier $(2h)^{-1}$). Those equations are solved separately and are not considered in this section. The third equation ($i = 3$) of the system (2.167) (under the additional condition that $m_i^+ = m_i^- = 0$) is the identity ($0 = 0$). From the first two equations we obtain

$$Q_{23} = -M_{11,1} - M_{12,2} - m_1, \\ Q_{31} = -M_{21,1} - M_{22,2} - m_2. \tag{2.170}$$

The substitution of these expressions into the third one of the system (2.166) leads to the equilibrium equation:

$$M_{2\alpha,\alpha 1} - M_{1\alpha,\alpha 2} - m_{2,1} - m_{1,2} + q_3 = 0. \tag{2.171}$$

With the hypotheses of Kirchhoff–Love we obtain the following relations of the displacement $w = w(x, y)$ of the middle plane $z = 0$ with the components of the strain tensor:

$$\varepsilon_{11} = -zw_{,11}, \qquad \varepsilon_{22} = -zw_{,22}, \\ \varepsilon_{12} = -zw_{,12}, \qquad \varepsilon_{33} = -\nu(\varepsilon_{11} + \varepsilon_{22})/(1 - \nu). \tag{2.172}$$

We use the notation $w_{,ij} = \partial^2 w/\partial x_i \partial x_j$.

The substitution of the expression (2.172) into the Hooke law (remember that we are dealing with isotropic material) and substitution of the calculated stresses into the formulae for the components $M_{\alpha\beta}$ of the moment tensor, give the following result:

$$M_{11} = -M_{22} = \frac{2Eh^3}{3(1+\nu)} w_{,12}, \quad M_{12} = \frac{2Eh^3}{3(1-\nu^2)}(w_{,22} + \nu w_{,11}), \\ M_{21} = -\frac{2Eh^3}{3(1-\nu^2)}(w_{,11} + \nu w_{,22}). \tag{2.173}$$

Finally, the substitution of (2.173) into the equation (2.171) leads to the well-known equation from the Sophie Germain classical theory of thin plate bending:

$$\varDelta\varDelta w = -\frac{q_3}{D} \equiv \frac{q}{D}, \tag{2.174}$$

where $D = 2Eh^3/3(1 - \nu^2)$ is the bending stiffness of the membrane.

To the equation (2.174) we have to add the boundary conditions. The basic types of boundary conditions are

1. Clamped boundary

$$w|_\varGamma = 0, \quad \left.\frac{\partial w}{\partial \nu}\right|_\varGamma = 0; \tag{2.175}$$

2. Hinge support

$$w|_\varGamma = 0, \quad M_\tau|_\varGamma = (-M_1\nu_2 + M_2\nu_1)|_\varGamma = 0, \tag{2.176}$$

   where $\nu = (\nu_1, \nu_2)$ is the outward normal to the curve $\varGamma$ of the domain $\varLambda$, occupied by the middle plane of the membrane in the plane Oxy, $M_\alpha = M_{\alpha\beta}\nu_\beta$;

3. Free boundary
$$M_\tau|_\Gamma = 0, \quad \left(R_3 - \frac{\partial M_\nu}{\partial \nu}\right)\bigg|_\Gamma = 0, \tag{2.177}$$
where R_3 is calculated according to the formula (2.164), $M_\nu = M_1\nu_1 + M_2\nu_2$.

If we solve the problem for the displacement w, then in the equations (2.176) and (2.177) the forces M_1, M_2, and R_3 must be written through the displacement. The derivation of the second condition in (2.177) belongs to Kirchhoff. In principle, instead of zeros on the right-hand sides of equations (2.175)–(2.177), we can have prescribed functions.

Stability equation and the bending membrane equation

Let a plate be loaded, beside the normal load $q_3(x,y)$, by the forces $q_1(x,y)$, $q_2(x,y)$ and, on the part Γ_σ of the curve Γ, by the forces $P_1(x,y)$, $P_2(x,y)$, $(x,y) \in \Gamma_\sigma$. The problem is to construct a mathematical model which takes into account the action of the internal stresses $\sigma^0_{\alpha\beta}$ in the middle plane (generated by the external forces $q_\alpha(x,y)$, $P_\alpha(x,y)$) on the deflection $w(x,y)$.

We build a model in which the inverse influence of the deflection w on the stresses $\sigma^0_{\alpha\beta}$ is negligible. Choose, as before, an arbitrary subdomain in the domain Ω, being a cylinder with the lateral surface parallel to the axis Oz. Write the conditions of its equilibrium, taking into the account the projection $(t^0_\alpha)_3$ on the axis Oz of the vector
$$t^0_\alpha = \sigma^0_{\alpha\beta}\nu_\alpha \tag{2.178}$$
of the surface forces on the cylinder lateral boundary. These projections appear due to the rotation of the normal ν to the lateral boundary under the deflection w and are equal to
$$(t^0_\alpha)_3 = t^0_\alpha \frac{\partial w}{\partial x^\alpha} = \sigma^0_{\alpha\beta}\nu_\beta w_{,\alpha}. \tag{2.179}$$
Using the same reasons as for the derivation of the Sophie Germain equation, we obtain the equation
$$\Delta\Delta w - \frac{2h}{D}(\sigma^0_{\alpha\beta} w_{,\alpha})_{,\beta} = \frac{q}{D}, \tag{2.180}$$
which is called the *stability equation.* This name is due to the fact that the BVPs for the equation (2.180) can have nonzero solutions even when $q = 0$ and boundary conditions for w are homogeneous. Physically, it is explained by the fact that the membrane, compressed by the forces, parallel to its middle plane, can have a curvilinear equilibrium form. Transition from the flat equilibrium state (where $w = 0$) to the curvilinear one ($w \neq 0$) under the same external load is called *loss of stability* (*buckling*). If we neglect the second term on

the left-hand side of the equation (2.180), then we again obtain the Sophie Germain equation.

It is possible to use the other limit transition, when the stiffness of the membrane D tends to zero. For this limit case, from the equation (2.180), we have the equation

$$(\sigma^0_{\alpha\beta} w_{,\alpha})_{,\beta} = q/2h. \tag{2.181}$$

In the traditional mathematical formulation of the membrane deflection it is assumed that

$$\sigma^0_{\alpha\beta} = \delta_{\alpha\beta} T, \quad T = \text{const} > 0, \tag{2.182}$$

where the value T is the tension of the membrane. Substitution of (2.182) into (2.181) leads to the Poisson equation for $w(x, y)$

$$\Delta w(x, y) = q/(2hT) \equiv f(x, y), \tag{2.183}$$

called the *membrane bending equation*. We complement it by either the Dirichlet boundary condition

$$w|_\Gamma = g(x, y), \quad (x, y) \in \Gamma, \tag{2.184}$$

or the Neumann boundary condition

$$\left.\frac{\partial w}{\partial \nu}\right|_\Gamma = h(x, y), \quad (x, y) \in \Gamma, \tag{2.185}$$

or a combination of these two boundary conditions.

In the end, we notice that equation (2.183) allows us to describe the deformation of structural elements such as thin soft membranes manufactured from tissues, thin film, etc. In this theory, a new problem appears related to the possible breaking of the condition $T > 0$. This breaking leads to the phenomenon of the suspending of the membrane. On the border of the suspending domain, one has to pose the free condition $\partial w/\partial \nu = 0$, where the suspended domain is unknown *a priori* and has to be found while solving the problem. Such a problem is, in general, nonlinear and its solution requires some special methods.

2.3.6 Transition to the variational formulation

The most important elasticity problems reduce to three basic types of equations: the Poisson equation or its special case, the Laplace equation, the biharmonic equation (2.147) or (2.174), and a general system of partial differential equations (2.123).

As we have seen, even for a membrane, there are various forms of governing equations, and various possible boundary conditions. This means that there are many possible variational formulations of the problems. We approach the problems in a general formulation.

Dirichlet and Neumann problems for the Poisson equation

We start with the Dirichlet problem for the Poisson equation in n-dimensional space:

$$-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}\right)\equiv -\Delta u = f(x), \tag{2.186}$$

$$u|_{\Sigma}=0, \quad \Sigma\equiv\partial\Omega. \tag{2.187}$$

The space V where we search to the solution of the variational problem is the Hilbert space $H_0^1(\Omega)$ (see (1.57)). By supposition, the boundary of the domain $\Omega\subset\mathbb{R}^n$ has the Lipschitz property (see Definition 1.65).

So, let $v\in H_0^1(\Omega)$. Multiply the left- and right-hand sides of the equation (2.186) by the element v and integrate the result over the domain Ω:

$$-\int_\Omega \Delta u\, v\, d\Omega = \int_\Omega f\, v\, d\Omega. \tag{2.188}$$

Apply the Green formula to the left-hand side:

$$\int_\Omega (-\Delta u) v\, d\Omega = \int_\Omega \operatorname{grad} u\cdot \operatorname{grad} v\, d\Omega - \int_\Sigma v\frac{\partial u}{\partial \nu}\, d\Sigma. \tag{2.189}$$

Using the denotations

$$a(u,v)=\int_\Omega \operatorname{grad} u\cdot\operatorname{grad} v\, d\Omega, \quad \langle f,v\rangle=\int_\Omega fv\, d\Omega \tag{2.190}$$

and the boundary condition (2.187), we conclude that any solution of the problem (2.186) and (2.187) satisfies the variational equation

$$a(u,v)=\langle f,v\rangle \quad \forall v\in V\equiv H_0^1(\Omega), \tag{2.191}$$

sometimes called the *integral identity*. Notice that in the traditional setting of the problem instead of $v\in V$ one uses the variation $\delta u=v-u$ of the solution, which also belongs to the space V. Instead of the variational equation (2.191) we will have

$$a(u,\delta u)=\langle f,\delta u\rangle \quad \forall \delta u = v-u,\ v\in V,\ u\in V.$$

The solution of the variational equation (2.191) is called the generalized solution of the problems (2.186) and (2.187). If the solution u of this equation has the second-order derivatives then, using the formula (2.189) for the transformation of the left side of (2.190), we obtain the equation

$$\int_\Omega (-\Delta u - f)v\, d\Omega = 0 \quad \forall v\in V. \tag{2.192}$$

From this equation, we obtain the equation (2.186) due to the density of V in the space $H^2(\Omega)$. Thus, any generalized twice differentiable solution satisfies the local equation (2.186) the and boundary condition (2.187).

We now construct the equation analogous to (2.17). For this purpose, introduce the functional

$$J(v) = \frac{1}{2} \int_\Omega |\operatorname{grad} v|^2 \, d\Omega. \tag{2.193}$$

This functional is the bending energy for the membrane problem and the energy of torsion for a bar torsion problem.

The total energy of the system is defined by

$$\Pi(u) = J(u) - \langle f, u \rangle. \tag{2.194}$$

Using this definition, rewrite the equation (2.191) as

$$\delta[J(u) - \langle f, u \rangle] \equiv \delta \Pi(u) = 0. \tag{2.195}$$

Therefore, any solution of the problems (2.186) and (2.187) is the stationary point of the functional Π, and any stationary point of the functional Π, the element u, having the second-order derivatives, is the solution of the initial problem (2.186) and (2.187). Using Friedrichs inequality

$$\int_\Omega |\operatorname{grad} u|^2 \, d\Omega \geq \alpha \|u\|^2_{0,2,\Omega}, \quad u \in V, \ u|_{\partial\Omega} = 0, \ \alpha = \alpha(\Omega) > 0, \tag{2.196}$$

we can show that any stationary point of the functional Π is the minimum point and is unique (for more details see Chapter 3).

If the boundary condition (2.187) is nonhomogeneous, i.e.,

$$u|_\Sigma = g(x), \quad x \in \Sigma, \tag{2.187'}$$

all the previous statements can be demonstrated with some additional hypotheses for the function $g(x)$, the most important of which is that $g(x)$ is piece-wise smooth. To prove these statements, we introduce the substitution $u = u_g + \tilde{u}$, where u_g is an arbitrary function equal to $g(x)$ on the boundary and twice differentiable in Ω. In fact, it is sufficient to suppose that $u_g \in H^1(\Omega)$ and $g \in H^{1/2}(\Omega)$. The existence of such a function follows from the trace theorem (see Theorem 1.96).

For the difference $\tilde{u} = u - g$ we obtain the previous problems (2.186) and (2.187), where in (2.186) the right-hand side takes the form $f + \Delta u_g$. After transition to the minimization problem for the functional Π depending on the argument $\tilde{u}$, one uses the backward substitution $u = \tilde{u} - u_g$. Taking into account that the integral

$$\int_\Omega |\operatorname{grad} u_g|^2 \, d\Omega$$

is constant and disappears in the minimization procedure, we obtain the minimization problem of the functional (2.194), with the boundary condition on the admissible functions of type (2.187').

Now consider the Neumann problem for the Poisson equation

$$-\Delta u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{2.197}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Sigma} = 0. \tag{2.198}$$

The problems (2.197) and (2.198) is solvable only for functions f orthogonal (in $L^2(\Omega)$) to the unity function, i.e.,

$$\int_{\Omega} f\, d\Omega = 0. \tag{2.199}$$

The necessity of the condition (2.199) may be proved by the integration of the equation (2.197) over the domain Ω, with the Green formula (2.190) and the condition (2.198).

For transition to the variational problem, introduce the space $H^1(\Omega)/P_0$ quotient to $H^1(\Omega)$ with respect to polynomials degree 0 (see Definition 1.79).

Repeating the argument used for the Dirichlet problem, we may establish that any solution of the problems (2.197) and (2.198) will satisfy the variational equation

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V \tag{2.200}$$

or the equation

$$a(u, \delta u) = \langle f, \delta u \rangle \quad \forall \delta u = v - u,\ v \in V,\ u \in V. \tag{2.200'}$$

with the previous notation (2.190). Assuming that the solution u of the equation (2.200) has the second-order derivatives, we find that the element u satisfies the equation (2.197) and the boundary condition (2.198). Indeed, using the Green formula for the left-hand side of (2.200), we obtain

$$\int_{\Omega} (-\Delta u - f) v\, d\Omega + \int_{\Sigma} \frac{\partial u}{\partial \nu} v\, d\Sigma = 0 \quad \forall v \in V. \tag{2.201}$$

Let the function $v \in D(\Omega)$. Using the density of the set $D(\Omega)$ in V, we arrive at the equation (2.197). From the equations (2.197) and (2.201) we obtain

$$\int_{\Sigma} \frac{\partial u}{\partial \nu} v\, d\Sigma = 0 \quad \forall v \in V, \tag{2.202}$$

and, hence, the condition (2.198) follows.

Thus, we conclude that the Neumann boundary condition (2.198) is the *natural* one, i.e., the exact solution of the variational problem satisfies the condition (2.198) automatically.

The transition from the equation (2.200) to the minimization problem for the functional Π equal to the potential energy of the system

$$\Pi = 1/2a(v,\, v) - \langle f, v\rangle \tag{2.203}$$

on the space V. The proof of the uniqueness of the minimum point, is performed using the Poincaré inequality (see (1.55)).

If the boundary condition (2.198) is nonhomogeneous, i.e.,

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Sigma} = h(x), \quad x \in \Sigma, \tag{2.198'}$$

then, repeating the above reasoning, we establish that, to be able to solve the problem, the equality

$$\int_{\Omega} f\, d\Omega + \int_{\Sigma} h\, d\Sigma = 0 \tag{2.204}$$

must hold and that the initial problem is equivalent (assuming the existence of the second-order derivatives of the variational equation) to the variational equation

$$\int_{\Omega} \operatorname{grad} u \cdot \operatorname{grad} v\, d\Omega = \int_{\Omega} fv\, d\Omega + \int_{\Sigma} hv\, d\Sigma \quad \forall v \in V \tag{2.205}$$

in the space V, being the space quotient to $H^2(\Omega)$ with the respect to the polynomials $P \in P^1$ which describe all the motions of the body Ω as a rigid, i.e., $p \in P \Rightarrow p = c_1 + c_2 \times x$, $c_1 = \text{const}$, $c_2 = \text{const}$. The equation (2.205) is equivalent to the minimization problem for the functional

$$\Pi(v) = \frac{1}{2}a(v,\, v) - \langle f, v\rangle - \int_{\Sigma} hv\, d\Sigma \tag{2.206}$$

in the same space V. This statement can be proved, as before, by using the inequality (2.206).

Dirichlet problem for the biharmonic equation

Consider the problem

$$\Delta\Delta u = f(x, y), \quad (x, y) \in \Lambda \subset \mathbb{R}^2, \tag{2.207}$$

where Λ is a domain in $\mathbb{R}^2$, with the boundary $\Gamma = \partial\Omega$. Notate by ν the unit outward vector orthogonal to Γ.

The boundary conditions

$$u|_{\Gamma} = 0, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} = 0 \tag{2.208}$$

follow from the problems (2.147) and (2.148) by the selection of a function satisfying the conditions (2.148) and the subtraction of this function from the initial solution $U(x, y)$. The functional space V, in which we search for the solution, is the space $H_0^2(\Lambda)$, (see the definition (1.57)).

We use now the Green formula for the biharmonic operator:

$$\int_\Lambda (\Delta\Delta w) v \, d\Lambda = \int_\Lambda [w_{,xx} v_{,xx} + 2w_{,xy} v_{,xy} + w_{,yy} v_{,yy}] \, d\Lambda + \int_\Gamma \Big\{ v[(w_{,xxx} + w_{,xyy})\nu_1 + (w_{,xxy} + w_{,yyy})\nu_2] - v_{,x} \left(\frac{\partial w}{\partial \nu}\right)_{,x} - v_{,y} \left(\frac{\partial w}{\partial \nu}\right)_{,y} \Big\} ds. \tag{2.209}$$

Let u be the solution of the problems (2.207) and (2.208). Multiply the equation (2.207) by an arbitrary element $v \in V$, integrate the result over the domain Λ and use the formula (2.209) with the boundary conditions (2.208). The result is the variational equation

$$\int_\Lambda (u_{,xx} v_{,xx} + 2u_{,xy} v_{,xy} + u_{,yy} v_{,yy} - fv) \, d\Lambda = 0 \quad \forall v \in V. \tag{2.210}$$

Introducing the notation

$$a(u, v) = \int_\Lambda (u_{,xx} v_{,xx} + 2u_{,xy} v_{,xy} + u_{,yy} v_{,yy}) \, d\Lambda, \tag{2.211}$$

$$\langle f, v \rangle = \int_\Lambda f v \, d\Lambda,$$

we rewrite the equation (2.210) in the standard form

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V, \; u \in V, \tag{2.212}$$

to which all the linear problems have been transformed. These equations (see (2.42), (2.65), (2.200), and (2.212)) have common properties which provide for solvability and consist of symmetric positive definiteness of the bilinear form $a(u, v)$ and of the continuity of the linear form $\langle f, v \rangle$ on the space V.

In Chapter 3, we will demonstrate that the symmetry of the functional $a(u, v)$, the boundedness of the functional $\langle f, v \rangle$ and the positive definiteness of the functional $a(u, v)$

$$a(v, v) \geq \alpha \|v\|_V^2, \quad \alpha = \text{const} > 0, \tag{2.213}$$

allow one to go from the equation (2.212) or from the equation in the variations

$$a(u, \delta u) = \langle f, \delta u \rangle \quad \forall \delta u = v - u, \; v \in V, \; u \in V \tag{2.214}$$

to the minimization problem for the functional

$$\Pi(v) = \frac{1}{2} a(v, v) - \langle f, v \rangle \tag{2.215}$$

equal to the potential energy of the system.

Considering the problem, we obtain

$$\Pi(v) = \frac{1}{2}\int_{\Lambda}[v_{,xx}^2 + 2v_{,xy}^2 + v_{,yy}^2 - 2fv]\, d\Lambda. \qquad (2.216)$$

Notice once more that the inequalities of positive definiteness (2.213) for the functionals $a(u, v)$ have not been proved so far. These proofs are based on the Sobolev embedding theorems and will be given for some particular cases later (in Section 2.4) after we have built the main variational settings for the linear problems.

As with the transition from the Dirichlet problem in the local settings (2.207) and (2.208) to the variational equation (2.210) and the problem of the minimization of the functional (2.216), one can also perform the corresponding transformations for the Neumann problems, where instead of the boundary conditions (2.208) for the equation (2.207) we pose the boundary conditions of type (2.176) or (2.177). These transformations are rather technical and present no new ideas with respect to the previous case, so we omit them.

Main boundary value problems for the 3D equations of linear elasticity theory

Consider the problem

$$-\frac{\partial}{\partial x_j}[a_{ijkl}\varepsilon_{kl}(u)] \equiv (\hat{A}\cdot u)_i = \rho F_i, \qquad (2.217)$$

$$u(x) = 0, \quad x \in \Sigma,\ \Sigma = \partial\Omega, \qquad (2.218)$$

where the whole of the boundary Σ of the domain Ω is fixed. The operator $\varepsilon_{kl}(u)$ is defined by the formula (2.115). The space V, in which we search for the solution of the variational problem, is the Hilbert space for the vector-functions $v = (v^1, v^2, \ldots, v^n)$, n being the dimensionality of the vector v, and $V = [H_0^1(\Omega)]^n$. For simplicity, we keep for V the notation $V = H_0^1(\Omega)$.

Using the symmetry (2.111) and (2.112) of the elasticity tensor a_{ijkl}, we establish the Green formula for the operator $\hat{A}$ of linear elasticity theory, by the formula (2.217)

$$\langle \hat{A}\cdot u,\, v\rangle = -\int_{\Omega}\frac{\partial}{\partial x_j}[\sigma_{ij}(u)]\nu_i\, d\Omega = -\int_{\Sigma}\sigma_{ij}(u)v_i\nu_j\, d\Sigma + \int_{\Omega}\sigma_{ij}(u)\varepsilon_{ij}(v)\, d\Omega, \qquad (2.219)$$

where in the linear theory $\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u)$.

Multiplying the first equations of the system (2.217) by the first component of the trial function $v \in V$, the second by the second component, etc., summing the results and integrating the sum over the domain Ω, using the Green formula (2.219) and taking into account the boundary condition (2.218), we obtain the variational equation

$$\int_{\Omega} a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(v)\, d\Omega = \int_{\Omega}\rho F\cdot v\, d\Omega \qquad (2.220)$$

or, in a shorter form,

$$a(u,v) = \langle \rho F, v\rangle \quad \forall v \in V,\ u \in V, \tag{2.221}$$

where

$$a(u,\,v) = \int_\Omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(v)\,d\Omega \tag{2.222}$$

is the symmetric bilinear form, $\langle \rho F, v\rangle$ is a linear form on space V, defined by the right-hand side of the variational equation (2.220).

In the classical variational settings of linear elasticity problems, instead of the element $v \in V$, one uses the variation $\delta u = v - u$, and instead of (2.221) one obtains the equation

$$a(u,\delta u) = \langle \rho F, \delta u\rangle \quad \forall \delta u = v - u,\ v \in V,\ u \in V, \tag{2.223}$$

called the *variational equation of Lagrange.* As in the previous problems, the solution of (2.221) (or (2.223)) is called the generalized solution of the problems (2.217) and (2.218). Any generalized solution having the second-order derivatives satisfies the local equation and conditions (2.217) and (2.218).

Introduce the functional (deformation energy)

$$J(v) = \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{kl}(v)\varepsilon_{ij}(v)\,d\Omega \equiv \frac{1}{2}a(v,\,v), \tag{2.224}$$

and the functional Π (the total energy of the system)

$$\Pi(v) = J(v) - \langle \rho F, v\rangle. \tag{2.225}$$

Then the equation (2.223) can be rewritten in the form

$$\delta[J(u) - \langle \rho F, u\rangle] = \delta\Pi(u) = 0, \tag{2.226}$$

from which it follows that any solution of the problems (2.217) and (2.218) is the *stationary point of functional* Π. Any stationary point of the functional Π is a generalized solution of the problems (2.217) and (2.218).

Introduce the hypothesis about the positive definiteness of the functional $a(v,\,v)$,

$$a(v,\,v) \geq \alpha\|v\|_V^2, \quad \alpha = \text{const} > 0, \tag{2.227}$$

the background for which will be given below. Then we can demonstrate that any stationary point the of functional Π is the minimum point, and that the minimum point, corresponding to the given external forces, is unique.

The mixed problem of linear elasticity theory

$$(\hat{A}\cdot u)_i = \rho F_i, \tag{2.228}$$

$$u(x) = 0, \qquad x \in \Sigma_u \subset \Sigma, \tag{2.229}$$

$$\sigma_{ij}\nu_j|_{\Sigma_\sigma} = P_i, \qquad \Sigma_u \cup \Sigma_\sigma = \Sigma \tag{2.230}$$

can be transformed in the same way with the following difference:

- In the definition of the space V, instead of (2.218) we use the condition (2.229).
- Instead of the variational equation (2.221), we have the equation

$$a(u, v) = L(v) \quad \forall v \in V,\ u \in V, \tag{2.231}$$

where $L(v)$ denotes the linear functional

$$L(v) = \int_{\Omega} \rho F \cdot v \, d\Omega + \int_{\Sigma_\sigma} P \cdot v \, d\Sigma. \tag{2.232}$$

The stationary point of the functional $\Pi(u)$ which is the solution of the problem being considered has the form

$$\Pi(u) = 1/2a(u, u) - L(u). \tag{2.233}$$

If no part of the boundary Σ_u is fixed, then we must use the space V being the quotient to the space $H^1(\Omega)$ with respect to the polynomials which describe all the motions of the body Ω considered as a rigid. The uniqueness in V is demonstrated under the additional hypothesis that the resultant of the external forces and the resultant moment are both zero (see (2.120) and (2.121)). If this hypothesis does not hold, then we must solve a dynamic problem on the small deformation of the moving body Ω.

At the end of this section, we discuss alternative ways of constructing variational problems, which have been a source of many theories and methods in the mechanics of solids. Since in the definition of power in mechanics forces and displacements (velocities), being the generalized coordinates, are included symmetrically, then variation in the state of a mechanical system can be done not only by the use of infinitesimally small displacements under fixed forces, but also by variation of the internal forces under constant displacements.

Let us consider a 3D problem in linear elasticity theory (leaving 1D and 2D ones as exercises)

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i = 0, \tag{2.234}$$

$$\sigma_{ij} = a_{ijkl} \varepsilon_{kl}, \tag{2.235}$$

$$\varepsilon_{kl} = 1/2(u_{k,l} + u_{l,k}), \tag{2.236}$$

$$\sigma_{ij} \nu_j |_{\Sigma_\sigma} = P_i, \tag{2.237}$$

$$u|_{\Sigma_u} = g. \tag{2.238}$$

Let the true state of the equilibrium be described by the stresses σ_{ij}, strains ε_{ij}, and displacements u_i. Define a *statically admissible state* of the system as one for which the displacements and strains are those of the true equilibrium state, and the stresses differ from σ_{ij} by the infinitely small values $\delta\sigma_{ij}$. For the sum $\sigma_{ij} + \delta\sigma_{ij}$ the equations (2.234) and (2.237) holds. The variations $\delta\sigma_{ij}$ satisfy the homogeneous equilibrium equations

$$\delta\sigma_{ij,j} = 0 \tag{2.239}$$

and the homogeneous boundary conditions for the surface tractions

$$\delta\sigma_{ij}\nu_j|_{\Sigma_\sigma} = 0. \tag{2.240}$$

By supposition, the functions σ_{ij}, ε_{ij}, u_i in the systems (2.234)–(2.238) are independent of each other. Multiply each of the equation (2.236) by the corresponding stresses variation, sum and integrate over the domain Ω:

$$\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}\,d\Omega = \int_\Omega \delta\sigma_{ij}0,5(u_{i,j} + u_{j,i})\,d\Omega \quad \forall\delta\sigma_{ij} \in D(\Omega). \tag{2.241}$$

Using the symmetry of the tensor $\delta\sigma_{ij}$, we transform the right-hand part of the equation (2.241) using the Gauss formula. The result is the following:

$$\int_\Omega \varepsilon_{ij}\delta\sigma_{ij}\,d\Omega = \int_\Sigma \delta\sigma_{ij}u_i\nu_j\,d\Sigma - \int_\Omega \delta\sigma_{ij,j}u_i\,d\Omega. \tag{2.242}$$

Recall that the variations of the stresses satisfy the conditions (2.239) and (2.240). The right-hand part of the relation (2.242) is equal to

$$\int_{\Sigma_u} \delta\sigma_{ij}g_i\nu_j\,d\Sigma.$$

Using the equations (2.235) and (2.238), we establish that any solution σ_{ij} of the problems (2.234)–(2.238) satisfies the variational equation

$$-\int_\Omega A_{ijkl}\sigma_{kl}\delta\sigma_{ij}\,d\Omega + \int_{\Sigma_u} \delta\sigma_{ij}g_i\nu_j\,d\Sigma = 0, \tag{2.243}$$

where $\delta\sigma_{ij}$ is an arbitrary variation of stresses satisfying the conditions (2.239) and (2.240). A_{ijkl} is the compliance modulus tensor arising as the result of inversion of the relation (2.235), i.e.,

$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl}. \tag{2.244}$$

The variational equation (2.243) is called the *Castigliano variational equation*. It can be demonstrated that any solution of this equation satisfies (2.236) and the boundary condition (2.238), which now becomes the natural one. For any solution (2.243), the Saint-Venant condition (2.100) holds. A proof of this statement can be found, for example, in [Lov44]. It is omitted here in view of its awkwardness. It should be emphasized that for the initial (nonvaried) state the equilibrium equations (2.234) and the traction boundary conditions (2.237) have to hold.

It is easy to see that the variational equation (2.243) is the stationarity condition for the functional

$$J^*(\hat{\sigma}) = -\frac{1}{2}\int_{\Omega} A_{ijkl}\sigma_{kl}\sigma_{ij}\, d\Omega + \int_{\Sigma_u} \sigma_{ij} g_i \nu_j \, d\Sigma, \tag{2.245}$$

which is called the *Castigliano functional.* The positive definiteness of the compliance modulus tensor A_{ijkl},

$$A_{ijkl}\sigma_{kl}\sigma_{ij} \geq \beta\sigma_{ij}\sigma_{ij} \quad \forall\sigma_{ij} = \sigma_{ij},\ \beta = \text{const} > 0, \tag{2.246}$$

following from the positive definiteness of the elasticity modulus tensor, allows us to demonstrate immediately that any stationary point of the functional (2.245) is the maximum. This condition together with the linearity of constraints (2.234) and (2.237) on the varied functions allows us to prove the uniqueness of the maximum point corresponding to the prescribed external forces.

In Chapter 5, we shall show that the problem of the minimization of the functional (2.225) can be reduced to the problem of the maximization of the functional (2.245). Additionally, the relation between the character of the extremal points and the extremal values of these functionals will be established.

2.4 Positive definiteness of the potential energy of linear systems

2.4.1 Uniqueness of the minimum point

As shown in Section 2.3, the total potential energy of a linear deformed body has the form

$$\Pi(v) = 1/2a(v,v) - \langle f, v\rangle, \tag{2.247}$$

where $a(u,v)$ is a bilinear symmetric form and $\langle f, v\rangle$ is a linear form on the solution space V of the problem. Let the functional $a(v,v)$ be positive definite (see Definition 1.94):

$$a(v,v) \geq \alpha\|v\|_V^2 \quad \forall v \in V,\ \alpha = \text{const} > 0, \tag{2.248}$$

and the functional $\langle f, v\rangle$ be continuous, i.e., there exists a constant c, for which

$$\langle f, v\rangle \leq c\|f\|_{V^*}\|v\|_V. \tag{2.249}$$

V^* is the space of the linear functionals on V. The strict definition of the norm $\|f\|_{V^*}$ in this space is given in Section 1.2.8. We prove that the condition (2.248) implies the uniqueness of the solution of the minimization problem for the functional $\Pi(v)$ on the space V. To prove this statement, assume that there exist two different solutions $u_1 \neq u_2$:

$$\min_{v\in V} \Pi(v) = \Pi(u_1) = \Pi(u_2). \tag{2.250}$$

Consider the value

$$\Pi\left[\frac{1}{2}(u_1+u_2)\right]=\frac{1}{2}\Pi(u_1)+\frac{1}{2}\Pi(u_2)-\frac{1}{8}a(u_1-u_2,u_1-u_2). \qquad (2.251)$$

If $u_1 \neq u_2$, then using the property (2.248) from (2.250) and (2.251), we obtain

$$\Pi\left[\frac{1}{2}(u_1+u_2)\right]<\frac{1}{2}\Pi(u_1)+\frac{1}{2}\Pi(u_2)=\min_{v\in V}\Pi(v), \qquad (2.252)$$

which contradicts the condition (2.250); that is, the assumption $u_1 \neq u_2$ is not true.

As established in Section 3.3, the conditions (2.248) and (2.249) are sufficient for proving the existence of the solution. Thus, the most important characteristic of the linear systems is the positive definiteness inequality (2.248). Proofs of this inequality will be given below for a few particular cases.

2.4.2 Positive definiteness inequality for scalar functions

We start from the functional (2.45). Assuming the validity of the condition (2.49), one obtains the inequality

$$a(v,v)=\int_0^l ES(x)[v'(x)]^2dx \geq ES_0\int_0^l [v'(x)]^2dx. \qquad (2.253)$$

Notice that, using the boundary condition (2.28), we obtain

$$v(x)=\int_0^x v'(x)\,dx. \qquad (2.254)$$

Squaring the two sides of this equality and using the Cauchy–Buniakovskii inequality for the integrals, we find that

$$v^2(x)\leq x\int_0^x [v'(x)]^2dx \leq x\int_0^l [v'(x)]^2dx. \qquad (2.255)$$

Integrating the left- and right-hand sides of the resulting inequality over $[0,l]$, we obtain

$$\int_0^l v^2(x)\,dx \leq \frac{l^2}{2}\int_0^l [v'(x)]^2dx. \qquad (2.256)$$

Using the definition of the norm (2.37) in the space V where we look for the minimum of the functional (2.48), we rewrite the estimate (2.253) in the following form:

$$\begin{aligned} a(v,v) &\geq \frac{1}{2}ES_0\left[\int_0^l (v'(x))^2dx+\int_0^l (v'(x))^2dx\right] \\ &\geq \frac{1}{2}ES_0\min\{1,2/l^2\}\|v\|_V^2. \end{aligned} \qquad (2.257)$$

Using an analogous scheme, we can prove the positive definiteness of the functional (2.66). For the functional (2.193) in 2D problems one also can give the elementary proof of positive definiteness for the basic problems, Dirichlet, Neumann, and the mixed types [Mik64b]. To illustrate this, consider the Dirichlet problem in the domain $\Lambda \subset \mathbb{R}^2$ with the boundary Γ. Assume that the boundary condition is homogeneous. Include the domain Λ in a rectangle with the sides parallel to the coordinate axis $0 \leq x \leq a$, $0 \leq y \leq b$, and continue the solution $u(x,y)$ by zero onto the rest of the rectangle. Then, at any point (x_1, y_1) in the domain Λ the following representation (the analogue of (2.254)) holds:

$$u(x_1, y_1) = \int_0^{x_1} \frac{\partial u(x, y_1)}{\partial x}\, dx. \tag{2.258}$$

As in the derivation of (2.255), the Cauchy–Buniakovskii inequality for the integrals allows to obtain the estimate

$$u^2(x_1, y_1) \leq a \int_0^a \left[\frac{\partial u(x, y_1)}{\partial x}\right]^2 dx. \tag{2.259}$$

By integration of this estimate over the rectangle we obtain

$$\int_\Lambda u^2(x,y)\, d\Lambda \leq a^2 \int_\Lambda u_{,x}^2\, d\Lambda \leq a^2 \int_\Lambda \left(u_{,x}^2 + u_{,y}^2\right) d\Lambda. \tag{2.260}$$

Using the norms in the definition of the space $V = H_0^1(\Omega)$, one obtains

$$\begin{aligned} a(u,u) &= \frac{1}{2} \int_\Lambda \left(u_{,x}^2 + u_{,y}^2\right) d\Lambda \\ &= \int_\Lambda \left[\frac{1}{4}\left(u_{,x}^2 + u_{,y}^2\right) + \frac{1}{4}\left(u_{,x}^2 + u_{,y}^2\right)\right] d\Lambda \geq \\ &\geq \min\left\{\frac{1}{4}, \frac{1}{4a^2}\right\} \|v\|_V^2. \end{aligned} \tag{2.261}$$

The inequality (2.260) is called the *Friedrichs inequality.* This proof can easily be generalized to the case of any number of dimensions, and also to other problems containing the Laplace operator.

2.4.3 Applications to linear elasticity problems

The results (Theorem 1.67) cover the needs of most applied problems, but in problems of the mechanics of solids where combinations of derivatives appear, as in (2.98), specific problems arise in the demonstrations of positive definiteness. For the linear theory of elasticity these problems were solved first by Korn [Kor07]. The main result is known as *the Korn inequality*:

$$\int_\Omega \left[\varepsilon_{ij}(v)\varepsilon_{ij}(v) + v_i v_i\right] d\Omega \geq c\|v\|_V^2 \quad \forall v \in V,\ c = \text{const} > 0, \tag{2.262}$$

where V is the space of functions of type $(H^1(\Omega))^n$, $n = 2$ or 3 (see Definition 1.51). Since on the left-hand side of (2.262) we have some combinations of the first derivatives and on the right-hand side all the derivatives of the vector-functions v, the inequality (2.262) is far from obvious. The inverse inequality is trivial and, together with (2.262), it allows us to affirm that the norms on the left- and right-hand sides of (2.262) are equivalent.

There exist different proofs of the Korn inequality, see survey of the works on the subject in [Cia88]. Let us give here the most compact proof taken from [LVG02].

First, note that

$$\varepsilon_{ij}(v)\varepsilon_{ij}(v) = \frac{\partial v_i}{\partial x_i}\frac{\partial v_i}{\partial x_i} + \frac{1}{2}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{\partial v_i}{\partial x_j}^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right) \tag{2.263}$$

and

$$\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(v_i\frac{\partial v_j}{\partial x_i} - v_j\frac{\partial v_i}{\partial x_i}\right). \tag{2.264}$$

It follows from the Gauss formula that

$$\int_\Omega \sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) d\Omega = 0. \tag{2.265}$$

So

$$\int_\Omega \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\, d\Omega = \frac{1}{2}\int_\Omega \sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) d\Omega. \tag{2.266}$$

Using now the inequality (1.37), we obtain from (2.266) that

$$\int_\Omega \sum_{i,j=1}^{3}\left(\frac{\partial v_i}{\partial x_j}\right)^2 d\Omega \leq \text{const} \int_\Omega \sum_{i,j=1}^{3} \varepsilon_{ij}^2\, d\Omega. \tag{2.267}$$

For the completion of the Korn inequality proof we use the Friedrichs inequality in the form:

$$\int_\Omega v_i^2\, d\Omega \leq \text{const} \int_\Omega |\nabla v_i|^2\, d\Omega \tag{2.268}$$

and the positive definiteness of the elasticity modulus tensor a_{ijkl}:

$$a_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \geq \beta\varepsilon_{kl}\varepsilon_{kl} \quad \forall\varepsilon_{kl} = \varepsilon_{lk},\ \beta = \text{const} > 0, \tag{2.269}$$

which follows from the hypothesis on the positivity of the strain energy. Substituting the estimates (2.268) and (2.269) into (2.267), we obtain the Korn inequality (2.262).

Now, we prove that for the two main problems of the linear theory of elasticity with the boundary condition (2.218) or (2.237), where $\Sigma_\sigma = \Sigma$, the Korn inequality implies the positive definiteness of the functional (2.233).

From the definition (2.224) and the condition (2.269), it follows that

$$\int_\Omega a_{ijkl}\varepsilon_{kl}(v)\varepsilon_{ij}(v)\,d\Omega \geq \beta \int_\Omega \varepsilon_{kl}(v)\varepsilon_{kl}(v)\,d\Omega \tag{2.270}$$

and if we could establish that there is a positive constant c_0, for which

$$\int_\Omega \varepsilon_{kl}(v)\varepsilon_{kl}(v)\,d\Omega \geq c_0 \int_\Omega v_i v_i\,d\Omega. \tag{2.271}$$

Then the chain of equalities and inequalities

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,d\Omega = \frac{1}{2}\int_\Omega [\varepsilon_{ij}(v)\varepsilon_{ij}(v) + \varepsilon_{ij}(v)\varepsilon_{ij}(v)]\,d\Omega \geq$$
$$\geq \min\left\{\frac{1}{2}, \frac{c_0}{2}\right\}\left[\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,d\Omega + \int_\Omega v_i v_i\,d\Omega\right], \tag{2.272}$$

and the Korn inequality would imply the positive definiteness of the functional $a(v, v)$.

Consider a problem with a fully fixed boundary where the condition (2.218) holds. Let $v \neq 0$. Dividing the left- and the right-hand sides of the inequality (2.271) by the square of the norm v in $L_2(\Omega)$, we reduce the proof of this inequality on the whole space to that on the subset of functions satisfying the condition

$$\int_\Omega v \cdot v\,d\Omega = \int_\Omega v_i v_i\,d\Omega = 1. \tag{2.273}$$

We construct a proof by contradiction. We assume that there exists a sequence $\{v_n\}$ of elements $v_n \in V$ for which

$$\int_\Omega \varepsilon_{ij}(v_n)\varepsilon_{ij}(v_n)\,d\Omega \longrightarrow 0, \quad \int_\Omega v_{ni}v_{ni}\,d\Omega = 1, \quad n \to \infty. \tag{2.274}$$

The inequality (2.262) and the conditions (2.274) imply that the sequence $\{v_n\}$ is bounded in V. A bounded sequence in a Hilbert space is weakly compact. That means, from any bounded sequence in a Hilbert space, one can extract a subsequence, converging weakly to some element $v \in V$. The functional

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\,d\Omega$$

is weakly semicontinuous from above (see (3.56)). Then

$$\lim_{n\to\infty} \sup \int_{\Omega} \varepsilon_{ij}(v_n)\varepsilon_{ij}(v_n)\, d\Omega \geq \int_{\Omega} \varepsilon_{ij}(v)\varepsilon_{ij}(v)\, d\Omega. \tag{2.275}$$

Since from (2.274) it follows that the last integral (the underlining means the lower limit) is equal to zero, then $\varepsilon_{ij}(v) = 0$. From the boundary condition (2.218) it follows that $v = 0$. This conclusion contradicts the condition (2.273). Hence, the inequality (2.271), together with the positive definiteness of the functional $a(v, v)$, is proved.

Now consider the problem with the traction boundary condition

$$\sigma_{ij}\nu_j|_{\Sigma} = P_i \tag{2.276}$$

over the whole surface Σ of the domain Ω. Introduce the space $\tilde{V}$ quotient to $H^1(\Omega)$ with respect to a polynomials $P \equiv R$, as above. Recall that $\tilde{V}$ consists of the element of $\tilde{v}$ being a class of vector-functions. One function of a class differs from another function of the same class by a rigid displacement, i.e., the difference between two functions from the same class is equal to the vector

$$\rho = a \times x + b, \tag{2.277}$$

where a and b are constant vectors and $\times$ means the vector product. R, an element of the set R, has the form (2.277). More precisely,

$$R = \{\lambda\rho \mid \lambda \in \mathbb{R};\ \rho(x) = a \times x + b\}.$$

The positive definiteness of the form $a(\tilde{v}, \tilde{v})$ in the space $\tilde{V}$ means that the inequality

$$a(\tilde{v}, \tilde{v}) \geq \alpha\|\tilde{v}\|^2_{\tilde{V}}, \quad \alpha = \text{const} > 0 \tag{2.278}$$

holds. A proof of the inequality (2.278) is, e.g., in [DL72].

It follows from this result (see Chapter 3) that the solution of the variational equation

$$\int_{\Omega} a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(\delta u)\, d\Omega = \int_{\Omega} \rho F \cdot \delta u\, d\Omega + \int_{\Sigma} P \cdot \delta u\, d\Sigma \quad \forall \delta u = v - u \tag{2.279}$$

exists and it is unique in the space $\tilde{V}$, i.e., the stresses and strains are unique and the displacement field is a rigid body displacement field.

3

Variational Theory for Nonlinear Smooth Systems

3.1 Examples of nonlinear systems

Example 3.1. We consider a system with a finite number of degrees of freedom for which the state is defined by k generalized coordinates q_i and k generalized forces Q_i, respectively (Section 2.1). For equilibrium we have

$$Q_1\delta q_1 + Q_2\delta q_2 + \ldots + Q_k\delta q_k = 0. \tag{3.1}$$

The generalized forces depend nonlinearly on the coordinates, therefore the system is nonlinear. However, as was already mentioned in Section 2.1, the system can be conservative, i.e., the search for an equilibrium state can lead to the minimization of the function. Sufficient conditions for such an approach are known in classical analysis:

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial Q_j}{\partial q_i}, \quad \forall i, j. \tag{3.2}$$

Under these conditions the expression (3.1) is the total derivative of the force function U, defined by

$$U(q_1, \ldots, q_k) = \sum_{i=1}^{k} \int_{\widehat{AB}} Q_i(\xi_1, \ldots, \xi_k)\, d\xi_i, \tag{3.3}$$

where the integral on the right-hand side is an integral on a line. It is taken along the curve $\widehat{AB}$ connecting the points A and B in the k-dimensional space of generalized coordinates. The point B has the coordinates $(q_1, \ldots, q_k)$ and A has the coordinates $(q_{10}, \ldots, q_{k0})$. The choice of A and the curve $\widehat{AB}$ does not affect the generalized forces or the state of equilibrium.

Example 3.2. For the rod problem in Section 2.2 two kinds of nonlinearity are possible. First, the stress σ may depend nonlinearly on the longitudinal strain $\varepsilon = du/dx$:

$$\sigma = \phi(\varepsilon). \tag{3.4}$$

Secondly, the external forces F may depend nonlinearly on the unknown function $u(x)$, i.e.,

$$F = F(x, u(x)). \tag{3.5}$$

Instead of (2.27) we obtain the equation

$$-\frac{d\phi(u')}{dx} = F(x, u(x)). \tag{3.6}$$

The variational equation for the equilibrium under the boundary conditions (2.28) and (2.29) is as follows:

$$\int_0^l \phi(u'(x))\delta u'(x)\,dx = \int_0^l F(x, u(x))\delta u(x)\,dx \quad \forall \delta u = v - u,\ v \in V,\ u \in V. \tag{3.7}$$

The definition of the space V is given in Section 2.2.

Let us find the functional $J(u)$ such that the variational equation (3.7) takes the form of the equation $\delta J(u) = 0$. We assume that the functional J has the form

$$J(u) = \int_0^l \Phi(u'(x))\,dx - \int_0^l \Psi(x, u(x))\,dx. \tag{3.8}$$

Writing the stationary conditions of the given functional in the form

$$\delta J(u) = 0 \tag{3.9}$$

and recalling the definition of the variation of the functional

$$\delta J(u) = \lim_{t\to 0} \frac{1}{t}[J(u + t(v - u)) - J(u)],$$

we obtain

$$\int_0^l \frac{d}{dt}\Phi(u' + t(v' - u'))|_{t=0}\,dx - \int_0^l \frac{d}{dt}\Psi(x, u + t(v - u))|_{t=0}\,dx = 0. \tag{3.10}$$

We look for the stationary point of the functional $J(u)$ in the form $u_0 + \tau(v - u_0)$, where u_0 is an arbitrary element of the space V and $\tau \in [0, 1]$. We assume that the solution of the equation (3.7) has this form. Then, we obtain the equality

$$\int_0^l \phi(u_0' + \tau(v' - u_0'))(v' - u_0')\,dx - \int_0^l F(x, u_0 + \tau(v - u_0))(v - u_0)\,dx$$
$$= \int_0^l \frac{d}{dt}\Phi(u_0' + \tau(v' - u_0'))\,dx - \int_0^l \frac{d}{dt}\Psi(x, u_0 + \tau(v - u_0))\,dx. \tag{3.11}$$

Integrating both sides by τ from 0 to 1, we find

$$\int_0^l \left[\int_0^l \phi(u_0' + \tau\delta u_0')\delta u_0' \, dx\right] d\tau - \int_0^l \left[\int_0^l F(x, u_0 + \tau\delta u_0)\delta u_0 \, dx\right] d\tau$$

$$= \int_0^l [\Phi(v') - \Psi(x, v)] \, dx - \int_0^l [\Phi(u_0') - \Psi(x, u_0)] \, dx$$

$$\equiv J(v) - J(u_0). \tag{3.12}$$

Consequently, we can see that the functional J can be recovered by the equation (3.7) apart from an arbitrary constant $J(u_0)$. Choose $u_0 = 0$, and suppose that $J(0) = 0$ at the point $u_0 = 0$. Then the functional $J(v)$ has the following form:

$$J(v) = \int_0^l \left[\int_0^l \phi(\tau v'(x))v'(x) \, dx\right] d\tau - \int_0^l \left[\int_0^l F(x, \tau v(x))v(x) \, dx\right] d\tau. \tag{3.13}$$

In principle, other nonlinear problems can be considered similarly. To demonstrate rigorous analysis of such problems a brief summary of the theory is necessary, including the investigation of the existence and uniqueness of the solution.

3.2 Differentiation of operators and functionals

Consider a function space in which a norm is defined – a normed space. Suppose that the operation of the passage to a limit is defined (strong convergence), and all limit points belong to the space. The space is then a complete normed space, a Banach space [LVG02]. Suppose also that an inner product is defined, so that the space is a complete linear product space, a Hilbert space [LVG02].

Let $A : V \to H$ denote the operator acting from the Banach space V to the Banach space H. Assume that for some elements $u, \phi \in V$ there exists the finite limit

$$\lim_{t\to 0} \frac{1}{t}[A(u + t\phi) - A(u)] = A'(u, \phi). \tag{3.14}$$

This limit $A'(u, \phi)$ we call the *Gâteaux derivative* of the operator A at the point u in the direction ϕ.

If the limit $A'(u, \phi)$ exists for every element $\phi \in V$ and $\phi \rightarrowtail A'(u, \phi)$ is a *linear continuous operator*, then $A'(u, \phi)$ is called the *Gâteaux derivative* of the operator A at the point u.

Let $\mathbb{R}^n$ denote n-dimensional Cartesian space, and $\mathbb{R}^1 \equiv \mathbb{R}$ denote the space of real numbers. Let, as earlier, $A : V \to \mathbb{R}$. Then the operator A is called a *functional*.

If $A : V \to \mathbb{R}$ has the form

$$J(v) = 1/2a(v,v) \in \mathbb{R}, \tag{3.15}$$

where $a(u,v)$ is a bilinear symmetrical form defined on V, then

$$A'(u,\phi) = a(u,\phi). \tag{3.16}$$

If

$$J(v) = \int_\Omega g(v(x))\, d\Omega,$$

where $g(v)$ is a differentiable function in the usual sense for which all the above operations are valid and Ω is a domain in $\mathbb{R}^n$, then

$$J'(u,\phi) = \int_\Omega \frac{dg(u(x))}{du}\phi(x)\, d\Omega.$$

We shall use the notation J for functionals. (In some cases we shall also use other notations.)

Let us assume that for some elements u, ϕ, ψ the finite limit

$$\lim_{t\to 0} \frac{1}{t}[A'(u+t\psi,\phi) - A'(u,\phi)] \equiv A''(u,\phi,\psi) \tag{3.17}$$

exists. We call this limit the *second Gâteaux derivative* of operator A at the point u in the direction ϕ, ψ.

If the limit $A''(u,\phi,\psi)$ exists for all ϕ, ψ then the operator A is called *twice differentiable at the point u*. We note that this definition implies the linearity and, in particular, homogeneity of the second Gâteaux derivative, i.e.,

$$A''(u,\lambda\phi,\mu\psi) = \lambda\mu A''(u,\phi,\psi) \quad \forall\lambda \in \mathbb{R},\ \forall\mu \in \mathbb{R}. \tag{3.18}$$

It is easy to calculate the second Gâteaux derivative of the symmetric bilinear form by the formula

$$J''(u,\phi,\psi) = a(\phi,\psi).$$

The higher-order derivatives can be defined in the same manner.

Let the functional $G : V \to \mathbb{R}$ be differentiable in the Gâteaux sense at each point of the interval $u + t\phi$, $t \in [0,1]$, in the direction ϕ. Then there exists number $t_0 \in [0,1]$, for which the equality

$$J(u+\phi) = J(u) + J'(u+t_0\phi,\phi) \tag{3.19}$$

holds. This relation is called the *finite increment formula* of the functional. To verify the formula, define the function $f(t) = J(u+t\phi)$, for which we have:

$$\frac{df}{dt} \equiv f'(t) = J'(u+t\phi,\phi). \tag{3.20}$$

The finite increment formula of the function reads

$$f(1) = f(0) + f'(t_0). \tag{3.21}$$

Together with the equality (3.20), this leads to the relation (3.19).

Similarly, we can establish an analog of the Taylor formula for a functional

$$J(u+\phi) = J(u) + J'(u,\phi) + \frac{1}{2}J''(u+t_0\phi,\phi,\phi). \tag{3.22}$$

Under suitable assumptions of differentiability, the Taylor formulae of higher order can be introduced as well.

For an arbitrary operator, these formulae are usually not true. However, if we define the space of linear functionals H^* on H, where H is the range of A, and denote the value of the element $g \in H^*$ for the element $u \in H$ by $\langle g, u\rangle$, then for the functional $\langle g, u\rangle$ the following formulae hold:

$$\langle g, A(u+\phi)\rangle = \langle g, Au\rangle + \langle g, A'(u+t_0\phi,\phi)\rangle, \tag{3.23}$$

$$\langle g, A(u+\phi)\rangle = \langle g, Au\rangle + \langle g, A'(u,\phi)\rangle + \frac{1}{2}\langle g, A''(u+t_0\phi,\phi,\phi)\rangle, \tag{3.24}$$

where t_0 depends on g. The proofs of these relations are similar to those for functionals.

We now derive an inequality linking the norms of the range of values of an operator and the norms of its derivatives. Assuming the differentiability of the operator $A : V \to H$ at any point of the interval $u + t\phi$, $t \in [0, 1]$, we have the following inequality:

$$\|A(u+\phi) - A(u)\|_H \le \|A'(u+t_0\phi,\phi)\|_H, \tag{3.25}$$

where t_0 is some fixed number on the interval $[0, 1]$ which depends on ϕ also.

The proof is based on the Hahn–Banach theorem [Ped89]. Namely, there exists an element $g \in H^*$, $\|g\| = 1$ with the property

$$\|Au\|_H = \langle g, Au\rangle. \tag{3.26}$$

The norm of the linear operator is defined as the supremum of its range of value norms on the unit sphere [Ped89], for example,

$$\|g\|_{H^*} = \sup_{v\in H} \frac{\langle g, v\rangle}{\|v\|_H} = \sup_{v,\|v\|=1} \langle g, u\rangle.$$

Applying the Hahn–Banach theorem to the difference $A(u+\phi) - A(u)$, we obtain

$$\|A(u+\phi) - A(u)\|_H = \langle g, A(u+\phi) - A(u)\rangle. \tag{3.27}$$

Applying the finite increment formula (3.23) to the right side of (3.27) and using the inequality

$$|\langle g, A'(u+t_0\phi,\phi)\rangle| \le \|g\|_{H^*}\|A'(u+t_0\phi,\phi)\|_H,$$

we get the estimate (3.25).

In a similar way, in the case of twice differentiability we can demonstrate relation

$$\|A(u+\phi) - A(u) - A'(u,\phi)\|_H \le \frac{1}{2}\|A''(u+t_0\phi,\phi,\phi)\|_H, \tag{3.28}$$

where t_0 is some fixed number on the interval $[0,1]$, t_0 depends on ϕ and u.

Now we can give one of the main definitions of the theory under consideration, namely, the definition of the gradient of the functional $J : V \to \mathbb{R}$.

Let the Gâteaux derivative $J'(u,\phi)$ of the functional $J : V \to \mathbb{R}$ be linear and continuous with respect to the element ϕ. The element g from the space of the functionals V^* such that

$$J'(u,\phi) = \langle g, \phi\rangle \tag{3.29}$$

is called the *gradient of the functional* $J(u)$ *at the point* u and is denoted by $\operatorname{grad} J(u)$ or $\nabla J(u)$.

Example 3.3. Let

$$J(u) = \frac{1}{2}\int_0^1 (u'(x))^2 dx, \tag{3.30}$$

$$V = \Big\{ v \mid v = v(x),\ x \in (0,1);\ v(0) = 0,\ v(1) = 0;$$
$$(u,v) = \int_0^1 u'v' dx;\ v \in L^2(0,1),\ v' \in L^2(0,1) \Big\}. \tag{3.31}$$

It can be shown [Yos65] that $V^* = V$. Therefore, the linear functionals on V are the scalar product in V. Here, the Gâteaux derivative is

$$J'(u,\phi) = \int_0^1 u'(x)\phi'(x)\,dx = (u,\phi)_V = \langle u,\phi\rangle. \tag{3.32}$$

Comparing this result with the definition (3.29), we see that

$$\operatorname{grad} J(u) = u'(x). \tag{3.33}$$

If in the definition (3.31) the scalar product is replaced by the expression

$$(u,v) = \int_0^1 u(x)v(x)\,dx \tag{3.34}$$

then, integrating by part on the right-hand side of (3.32) and taking into account homogeneous boundary conditions, we arrive at the relation

$$J'(u,\phi) = -\int_0^1 u''(x)\phi(x)\,dx = -\langle u'',\phi\rangle, \tag{3.35}$$

from which it follows that

$$\operatorname{grad} J(u) = -u''(x). \tag{3.36}$$

Consequently, the form of the gradient depends on the space where it is defined.

Of course, usually in Banach spaces there is no scalar product. Only the norm is defined. The gradient in such spaces also can be defined because the concept of a linear functional makes sense. In fact, the calculation of the gradient is related to the solution of a certain extremum problem. Indeed, let

$$J'(u, \phi) = \langle J'(u), \phi \rangle. \tag{3.37}$$

We find

$$\sup_{\phi, \|\phi\|=1} J'(u, \phi) = \sup_{\phi} \frac{1}{\|\phi\|} J'(u, \phi) = \| \operatorname{grad} J(u) \|. \tag{3.38}$$

Let us denote by ϕ_e the element where the extremum is attained. Using the structure of the functional in the given Banach space, we find an element $g \in V^*$ for which the relation

$$J'(u, \phi_e) = \langle g, \phi_e \rangle \tag{3.39}$$

holds for all $\phi \in V$. This element g is the *gradient* of the functional J at the point u.

Let the functional $J : V \to \mathbb{R}$ be given, for which the second Gâteaux derivative $J''(u, \phi, \psi)$ is linear and continuous with respect to ϕ and ψ. The operator $H(u) : V \to V^*$, defined by the formula

$$J''(u, \phi, \psi) = \langle H(u)\phi, \psi \rangle \quad \forall \phi, \psi, \tag{3.40}$$

is called the *Hessian of the functional J at the point u*. (We will use the notation $H(u) = J''(u)$.)

Example 3.4. If

$$J(u) = \frac{1}{2} \int_0^1 (u'(x))^2 dx$$

then, with the definition (3.31) of the scalar product in the space V, operator $H(u)$ is the identity operator. If the definition (3.34) of the scalar product in V is used, then the H-operator will be the second derivative operator with a minus sign with respect to the argument x.

An important property of the functional $J : V \to \mathbb{R}$ is its convexity discussed in Chapter 1. Recall that a functional $J(v)$ is called *convex in the space V*, if for two arbitrary elements v_1, v_2 and any number $t \in [0, 1]$ the inequality

$$J((1-t)v_1 + tv_2) \leq (1-t)J(v_1) + tJ(v_2) \tag{3.41}$$

holds. If the inequality is strict for all $v_1 \neq v_2$ and all t such that $0 < t < 1$, then the functional $J(u)$ is called *strictly convex*.

This definition of (strict) convexity of the functional is a generalization of the concept of convexity downwards for functions with one and two variables. The latter has a simple geometrical interpretation: the chord with end points $(v_1, J(v_1))$ and $(v_2, J(v_2))$ does not lie below the graph of the function $J(v)$.

If the functional $J : V \to \mathbb{R}$ is Gâteaux differentiable, then the convexity property is equivalent to the inequality

$$J(v) \geq J(u) + J'(u, v-u) \quad \forall v, u \in V, \tag{3.42}$$

and strict convexity is equivalent to

$$J(v) > J(u) + J'(u, v-u) \quad \forall v, u \in V,\ u \neq v. \tag{3.43}$$

To prove this, we write the inequality (3.41) in the form

$$\frac{1}{t}[J(u + t(v-u)) - J(u)] \leq J(v) - J(u). \tag{3.44}$$

Calculating the limit for $t \to 0$, we obtain the inequality (3.42).

Conversely, assume that the inequality (3.42) holds. In this inequality, replace v by u and u by $u + t(v-u)$, respectively:

$$J(u) \geq J(u + t(v-u)) + J'(u + t(v-u), u - u - t(v-u)). \tag{3.45}$$

Replacing the element u by $u + t(v-u)$ in (3.42), we find

$$J(v) \geq J(u + t(v-u)) + J'(u + t(v-u), v - u - t(v-u)). \tag{3.46}$$

Due to the linearity of the functional $J'(u, \phi)$ w.r.t. ϕ, we can rewrite (3.45) and (3.46) in the form

$$\begin{aligned} J(u) &\geq J(u + t(v-u)) - tJ'(u + t(v-u), v-u), \\ J(v) &\geq J(u + t(v-u)) + (1-t)J'(u + t(v-u), v-u). \end{aligned} \tag{3.47}$$

Multiplying the first inequality in (3.47) by $1-t$, the second by t, and adding them, we arrive at the required inequality (3.42). The strict inequality (3.43) can be proved similarly.

If the functional $J : V \to \mathbb{R}$ is twice Gâteaux differentiable and

$$J''(u, \phi, \phi) \geq 0 \quad \forall u, \phi \in V, \tag{3.48}$$

then the functional J is convex on V. Moreover, if the relation

$$J''(u, \phi, \phi) > 0 \quad \forall u, \phi \in V,\ \phi \neq 0, \tag{3.49}$$

holds, then the functional J is strictly convex on V.

In order to prove this, we use the second-order Taylor formula (3.22)

$$J(v) = J(u) + J'(u, v-u) + \frac{1}{2}J''(u + t_0(v-u), v-u, v-u), \quad t_0 \in [0,1]. \tag{3.50}$$

Using the inequality (3.42) or (3.43), we obtain the inequality (3.48) or (3.49).

Example 3.5. Choose $J(v) = 1/2a(v,v) - \langle f, v\rangle$, where $a(v,v)$ is a bilinear form and $\langle f, v\rangle$ is a linear functional. If the form $a(v,v)$ is positive definite, i.e.,

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall v \in V, \ \alpha = \text{const} > 0, \tag{3.51}$$

then the strict convexity of the functional follows immediately from the formula $J''(u,\phi,\phi) = a(\phi,\phi)$.

Remark 3.6. For the analysis of the existence, regularity, etc., of the solution of the boundary problem, one sometimes uses (instead of the Gâteaux derivative) the *Frechet derivative*, denoted by $A'_{F(u,\phi)}$, which is defined for the operator $A : V \to H$ as a continuous and linear operator with respect to ϕ as the limit

$$\lim_{\phi \to 0} \frac{1}{\|\phi\|_V} \|A(u+\phi) - A(u) - A'_F(u,\phi)\|_H = 0. \tag{3.52}$$

The Frechet derivative is called a strong derivative, while the Gâteaux derivative is a weak derivative. Clearly, the existence of the Frechet derivative implies the existence of the Gâteaux derivative and they are equal. If the Gâteaux derivative exists, then for the existence of the Frechet derivative we require its continuity also [Ped89].

3.3 Existence and uniqueness theorems of the minimal point of a functional

Among the requirements which guarantee the existence of the minimal point, the most important is the condition of the weak lower or upper semicontinuity. In order to describe this property, we need some additional definitions.

We say that the sequence $\{v_n\}$ in the Banach space V is *weakly convergent* to the element $v \in V$ if

$$\lim_{n \to \infty} \langle f, v_n\rangle = \langle f, v\rangle \quad \forall f \in V^*. \tag{3.53}$$

Remember that V^* denotes the space of the linear, continuous functionals on V and $\langle f, v\rangle$ the value of the functional $f \in V^*$ on the element $v \in V$.

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. The point $x \in \mathbb{R}$ is said to be a condensation point of $\{x_n\}$, if there are an infinity of the sequence in any neighborhood of x, x may be in $\{x_n\}$. Then, there are an infinity of points of $\{x_n\}$ equal to x. Let $\varepsilon(x)$ be a set of the condensation points, and let the sequence $\{x_n\}$ be bounded above. The *upper limit of the sequence* $\{x_n\}$

$$\lim_{n \to \infty} \sup x_n \equiv \overline{\lim_{n \to \infty}}\, x_n \tag{3.54}$$

is the supremum of the number set $\varepsilon(x)$. If $\{x_n\}$ is bounded below, the *lower limit of the sequence* $\{x_n\}$

$$\lim_{n\to\infty}\inf x_n \equiv \varliminf_{n\to\infty} x_n \tag{3.55}$$

is the infimum of the number set ε.

The functional $J : V \to \mathbb{R}$ is called *weakly lower (upper) semicontinuous in* V if $J(v)$ is weakly semicontinuous from below (above) at any point of V. The functional $J : V \to \mathbb{R}$ is called *weakly semicontinuous from below (above) at the point* $u \in V$ if for arbitrary sequence $\{v_n\} \subset V$, being weakly convergent to the element u, the following inequality holds:

$$\lim_{n\to\infty}\inf J(v_n) \le J(u), \quad (\lim_{n\to\infty}\sup J(v_n) \ge J(u)). \tag{3.56}$$

Sometimes, in applications, the term lower (upper) semicontinuity is also used. In this case, in the above definitions, the weakly convergent sequences should be replaced by sequences that are convergent in the norm (strongly convergent sequences).

The functional $J(v)$ is strongly (weakly) continuous if it is strongly (weakly) continuous from above and below.

The direct verification of weak semicontinuity is a difficult task. Therefore, in practice, different criteria of weak lower semicontinuity are applied. The most important of these is the following.

Theorem 3.7 (Criterion of weak lower semicontinuity). *If the functional* $J : V \to \mathbb{R}$ *is convex and its first Gâteaux derivative* $J'(u,\phi)$ *is linear and continuous with respect to* ϕ*, then* $J(v)$ *is weakly lower semicontinuous.*

Proof. Choose any sequence $\{v_n\}$, weakly convergent in V to the element $u \in V$. The convexity implies that (see (3.42))

$$J(v_k) \le J(u) + J'(u, v_k - u). \tag{3.57}$$

Let us recover the definition (3.37) of the gradient $\nabla J(u)$ and rewrite the inequality (3.57) in the following manner:

$$J(v_k) \ge J(u) + \langle \nabla J(u), v_k - u\rangle. \tag{3.58}$$

Now, we use the linearity and the continuity of the mapping

$$\phi \longrightarrow \langle \nabla J(u), \phi\rangle \tag{3.59}$$

for the transition to the limit when $v_k \to u$ (weak convergence). In order to do this, we select some subsequence $\{v_{k_n}\} \subset \{v_k\}$, which is weakly convergent to the element u. Then, due to the linearity and continuity of the mapping (3.59), we have

$$\lim_{k_n\to\infty} J(v_{k_n}) \ge J(u) + \lim_{k_n\to\infty}\langle \nabla J(u), v_{k_n} - u\rangle = J(u). \tag{3.60}$$

The right-hand part of the relation (3.60) does not depend on the choice of the subsequence $\{v_{k_n}\}$. Therefore, the inequality holds also for the infimum for all possible subsequences $\{v_{k_n}\}$, i.e.,

$$\inf_{\{v_{k_n}\}} \lim_{k_n\to\infty} J(v_{k_n}) \equiv \underline{\lim}_{k\to\infty} J(v_k) \geq J(u).$$

We now consider problems of finding the minimum points of functionals not on the entire space V, but only on some subset $K \subset V$. We suppose that K is *convex* and *weakly closed*. We now define this term. K is convex if $u_1 \in K$, $u_2 \in K$ implies that every point on straight line joining u_1, u_2 is in K. The equation of the straight line between points u_1, u_2 has the form

$$v = (1-t)u_1 + tu_2. \tag{3.61}$$

Consequently, convexity means the following: if $u_1 \in K$, $u_2 \in K$ then

$$(1-t)u_1 + tu_2 \in K \quad \forall t \in (0,1). \tag{3.62}$$

The set $K \subset V$, where V is a Banach space, is called *weakly closed* if any sequence $\{v_k\} \subset K$ contains a subsequence $\{v_{k_n}\}$ which is weakly convergent to $u \in K$. (Recall that in the classical definition of closed set we have strong convergence.)

A point $\bar{u} \in K$ is called a relative (or local) minimum point (or, in brief, relative minimum point) of the functional $J : V \to \mathbb{R}$, if there exists a neighborhood $O(\bar{u})$ to the point $\bar{u}$, where the following inequality holds:

$$J(\bar{u}) \leq J(v) \quad \forall v \in K \cap O(\bar{u}). \tag{3.63}$$

The point $\bar{u} \in K$ is called the global minimum point of the functional $J(v)$ in the set K, if the inequality (3.63) is valid for all $v \in K$.

Further, we assume that there is no point in K where functional $J(v)$ is equal to $-\infty$ and we also exclude the trivial case $J(v) \equiv +\infty$.

Theorem 3.8 (Existence of global minimum). *Let $J : V \to \mathbb{R}$ be a weakly lower semicontinuous functional, and let the set $K \subset V$ be weakly closed and bounded and V be reflexive Banach space (that is, $V^{**} = V$). Then there exists at least one global minimum point $\bar{u}$ of the functional $J(v)$ in the set K.*

Proof. Let

$$l = \inf_{v\in K} J(v). \tag{3.64}$$

Let $\{u_k\}$ be a minimizing sequence, i.e.,

$$\lim_{k\to\infty} J(u_k) = l, \quad u_k \in K. \tag{3.65}$$

The existence of such a sequence follows from the definition of the infimum. Since the set K is bounded, the sequence $\{u_k\}$ is also bounded (in the norm of V). From the reflexivity of the Banach space V it follows that we can extract a weakly convergent subsequence from any bounded (in norm) sequence [Sob50]. Let us denote this subsequence by $\{u_{k_n}\}$ and its limit point by $\bar{u}$. Then, the property of weak closedness of K implies that $\bar{u} \in K$.

The weak lower semicontinuity permits us to write the following inequality:

$$J(u) \leq \lim_{k_n \to \infty} \inf J(u_{k_n}) \leq \lim_{k_n \to \infty} J(u_{k_n}) = l. \tag{3.66}$$

Since l is the infimum, it follows from (3.66) that

$$J(\bar{u}) \leq l. \tag{3.67}$$

Hence, l is finite and the element $\bar{u} \in K$ is the global minimum point of the functional $J(v)$ in the set K.

Theorem 3.9 (Existence of the global minimum point). *If all the assumptions of Theorem 3.8 are satisfied, except the boundedness of K, which is replaced by the condition*

$$\lim_{\|v\| \to \infty} J(v) = +\infty, \tag{3.68}$$

then there exists at least one global minimum point of the functional $J(v)$ on K.

Proof. Let us introduce the infimum l by the formula (3.64) and the minimizing sequence $\{u_k\}$ by (3.65). We state that this sequence (at least, beginning from some index) is bounded in the norm. Indeed, if we take the opposite assumption that there is no sequence $\{u_k\}$, for which the relation

$$\lim_{k_n \to \infty} \|u_{k_n}\| = l < +\infty, \tag{3.69}$$

holds, then, from the condition (3.68) and the definition (3.64), it follows that

$$J(v) \equiv +\infty. \tag{3.70}$$

This results in a contradiction with the supposition that the functional $J(v)$ is not identically equal to $+\infty$. This contradiction proves our statement about the boundedness of the sequence $\{u_k\}$. From this, the proof is the same as was given for Theorem 3.7, because, from some index, we have

$$\|u_k\| \leq M = \text{const} < +\infty \tag{3.71}$$

and, consequently, it is sufficient to find the minimum in a ball with radius M.

Important for the demonstration is the property (3.68) of the functional $J(v)$, which is also called *coercivity*. (The functional $J(v)$ is said to be coercive if the limit equality (3.68) holds.) Indeed, suppose that the functional is a function of a single variable $x \in \mathbb{R}$,

$$J(v) \equiv J(x) = \exp(x), \tag{3.72}$$

defined on the whole real axis $\mathbb{R}$. This functional does not have any global minimum point. We also notice that for quadratic functionals of the form (3.19) the condition (3.68) can be assured by the assumption of positive definiteness of the bilinear form $a(u, v)$.

Let us consider the basic problem: find the global minimum point $\bar{u}$ of the functional $J(v)$ in the set K. Briefly, we have the problem ($\wp$)

$$\inf_{v \in K} J(v). \tag{$\wp$}$$

For the point $\bar{u}$ we can formulate some statements which are collected in the following theorem.

Theorem 3.10 (Characteristic of the minimum point). *Assume that all the conditions of Theorem 3.7 (or Theorem 3.8) are satisfied and, moreover, that the functional $J(v)$ is convex and the set $K \subseteq V$ is convex. Then*

(i) Every local minimum point is also a global minimum point.
(ii) At the minimum point $\bar{u}$ the following inequality holds:

$$J'(\bar{u}, v - \bar{u}) \geq 0 \quad \forall v \in K. \tag{3.73}$$

(iii) If at a point $\bar{u} \in K$ the inequality (3.73) holds, then the element $\bar{u}$ is the minimum point of the functional $J(v)$ in the set K.
(iv) If $\bar{u}$ is an inner point (not on the boundary: use the notation $\bar{u} \in \operatorname{int} K$) of the set K, then at this point we have

$$J'(\bar{u}, \phi) = 0 \quad \forall \phi. \tag{3.74}$$

(v) If in our assumptions the requirement of convexity of the functional $J(v)$ is replaced by the condition of strict convexity, then the global minimum point is unique.

Proof. First we prove the statement (ii). From the definition of the point $\bar{u}$ it follows that in some neighborhood $O(\bar{u})$ the inequality

$$J(v) \geq J(\bar{u}) \tag{3.75}$$

holds. (The neighborhood $O(\bar{u})$ is defined using the norm in V.) Let us put into this formula $v = \bar{u} + t(v - \bar{u})$, $0 < t < 1$. Thus, $J(\bar{u} + t(\bar{v} - \bar{u})) - J(u) \geq 0$ for all $t > 0$. After substituting $J(\bar{u})$ on the left-hand side and dividing by the positive number t, we get

$$\frac{1}{t}[J(\bar{u} + t(\bar{v} - \bar{u})) - J(u)] \geq 0. \tag{3.76}$$

Letting $t \to 0+$, we arrive at the statement (ii).

Let us now prove statement (i). The convexity of the functional $J(v)$ implies the inequality

$$J(v) \geq J(\bar{u}) + J'(\bar{u}, v - \bar{u}) \quad \forall v \in K. \tag{3.77}$$

Using the demonstrated inequality (3.73) and assuming the existence of two relative minimum points u_1 and u_2, we set into (3.77) first $v = u_1$, $\bar{u} = u_2$, then $v = u_2$, $u = \bar{u}_1$, and arrive at the equality

$$J(u_1) = J(u_2), \tag{3.78}$$

which means that every local minimum point is the global minimum point.

To prove the statement (iii), we use the inequality (3.73) which has been demonstrated. Then this inequality and (3.77) (which is defined by convexity) together result in the required statement about the fact that point $\bar{u}$ is the minimum point.

Now we prove the statement (iv). Since $\bar{u} \in \operatorname{int} K$ then there exists $t > 0$ such that for all ϕ, $\bar{u} + t\phi \in K$ as well. So, the element

$$\bar{u} + t(v - \bar{u}) = (1 - t)\bar{u} + tv$$

is in K if $0 < t < 1$, i.e.,

$$\bar{u} \pm \phi \in K \quad \forall \phi = v - \bar{u},\ v \in K.$$

Substituting the combination $\bar{u} \pm \phi$ into the inequality (3.73), we have

$$\pm t J'(u, \phi) \geq 0 \quad \forall \phi, \tag{3.79}$$

which means that the equation (3.74) is valid.

Finally, in order to prove statement (v), we assume the opposite statement, i.e., there exist at least two different elements u_1 and u_2, where the minimum of the functional $J(v)$ is attained. From the proved statement (i) we have the equality

$$J(u_1) = J(u_2). \tag{3.80}$$

The assumption of the strict convexity of the functional $J(v)$ and the assumption that $u_1 \neq u_2$ imply the relation

$$J(u_2) > J(u_1) + J'(u_1, u_2 - u_1) \geq J(u_1). \tag{3.81}$$

Then from (3.73) and (3.81) it follows that

$$J(u_2) > J(u_1), \tag{3.82}$$

which contradicts the equality (3.80). Therefore, the problem of finding minimum point of the strictly convex functional $J(v)$ in the convex set K has a unique solution.

Theorems 3.8–3.10 and the results given in Section 2.4 mean that under conditions which guarantee the positive definiteness of the functional of potential energy, all the problems which have been considered earlier have a solution, and this solution is unique in the appropriate functional space.

Later, we will apply the theory to the analysis of more complicated nonlinear problems, for example, to problems with nonquadratic functionals, problems where the set of admissible solutions K differs from the functional space V on which the operators and functionals are defined. Before doing this, we examine the conditions which make it possible to transfer problems with operators to problems with functionals.

3.4 Condition for the potentiality of an operator

The variational equations considered in Chapter 2 were derived by means of integration by parts or with the more general Green formula and Gauss–Ostrogradski formulae. They can be rewritten as the operator equation:

$$\langle A(u), v\rangle = \langle f, v\rangle \quad \forall v \in V, \tag{3.83}$$

where the element f and the value $A(u)$ of the operator A on the element u belong to the dual space V^* (space of linear functionals). This statement will be written as

$$A : V \longrightarrow V^*. \tag{3.84}$$

Consider now the problem: find those conditions under which the operator (3.84) is potential, that is, there exists a functional $J : V \to \mathbb{R}$, such that the relation

$$A(u) = \nabla J(u) \tag{3.85}$$

holds. The following theorem gives the answer to this question [Vaĭ64].

Theorem 3.11 (Potentiality of an operator). *Assume that the operator (3.84) is Gâteaux differentiable, with the derivative $A'(u, \varphi)$, being a linear and continuous operator with respect to φ, at least in a ball S with the center point u_0. Moreover, it is also a continuous operator with respect to u in the same ball S. Then, for the potentiality of the operator A the following condition is necessary and sufficient:*

$$\langle A'(u, \varphi), \psi\rangle = \langle A'(u, \psi), \varphi\rangle \quad \forall \varphi, \psi \in V. \tag{3.86}$$

Proof. At first we prove the necessity of the condition (3.86), that is, we assume the existence of the functional $J : V \to \mathbb{R}$ with the property (3.85) such that the equality

$$\langle \nabla J(u), \varphi\rangle = \langle A(u), \varphi\rangle \quad \forall \varphi \tag{3.87}$$

holds. Then, we demonstrate that this equality implies the condition (3.86). Let us choose the point $u \in S$ and the elements $\varphi, \psi \in V$ with the unit norm $\|\varphi\| = 1$, $\|\psi\| = 1$, and select numbers a, b such that the relation

$$u + \alpha\varphi + \beta\psi \in S \quad \forall \alpha \in [0, a],\ \forall \beta \in [0, b] \tag{3.88}$$

holds. Consider the following combination:

$$\Delta(u) = J(u + a\varphi + b\psi) - J(u + a\varphi) - J(u + b\psi) + J(u). \tag{3.89}$$

Using the notation

$$\bar{\varphi}(u) = J(u + a\varphi) - J(u), \tag{3.90}$$

rewrite the quantity Δ in the form

$$\Delta(u) = \bar{\varphi}(u + b\psi) - \bar{\varphi}(u). \tag{3.91}$$

Using the Taylor formula and the homogeneity of the Gâteaux derivative with respect to the second argument, we obtain

$$\Delta = \bar{\varphi}'(u + \tau_1 b\psi, b\psi) = b[J'(u + a\varphi + \tau_1 b\psi, \psi) - J'(u + \tau_1 b\psi, \psi)], \tag{3.92}$$

where τ_1 is a positive number on the segment $[0, 1]$. Applying the definition of the gradient of the functional and the equality (3.87), we obtain

$$\Delta = b[\langle A(u + a\varphi + \tau_1 b\psi), \psi\rangle - \langle A(u + \tau_1 b\psi), \psi\rangle]. \tag{3.93}$$

Using the Taylor formula in the form (3.23), we have

$$\Delta = ab\langle A'(u + \tau_2 a\varphi + \tau_1 b\psi, \varphi), \psi\rangle, \tag{3.94}$$

where τ_2 is a positive number on the segment $[0, 1]$. Now, with the change of the roles of the functions φ and ψ, we obtain

$$\Delta = ab\langle A'(u + \tau_3 a\varphi + \tau_4 b\psi, \psi), \varphi\rangle, \tag{3.95}$$

where τ_3 and τ_4 are positive numbers on the segment $[0, 1]$.

Now equate the expressions (3.94) and (3.95) and divide both expressions by the product ab. Using the continuity of the derivative $A'(u, \varphi)$ with respect to the first argument, calculate the limits with $a \to 0$, $b \to 0$. We obtain the equality

$$\langle A'(u, \varphi), \psi\rangle = \langle A'(u, \psi), \varphi\rangle,$$

which completes the demonstration.

We now prove that the equality (3.86) assures the existence of the potential J of the operator A, i.e.,

$$A(u) = \nabla J(u). \tag{3.96}$$

Assume that the functional $J(u)$ exists. We first establish a connection between this functional and the operator A. The relation

$$\frac{d}{dt} J(u_0 + t(v - u_0)) = \langle A(u_0 + t(v - u_0)), v - u_0\rangle \tag{3.97}$$

holds for any u_0, v and numbers $0 \le t \le 1$. Integrate the equality (3.97) by parts with respect to t from 0 to 1 to see that

$$J(v) - J(u_0) = \int_0^1 \langle A(u_0 + t(v - u_0)), v - u_0 \rangle \, dt. \tag{3.98}$$

Note that this formula is the generalization of that which was obtained for the particular case (3.13). We also note that the existence of the integral on the right-hand side in (3.98) is provided by the continuity of the derivative $A'(u, \varphi)$ with respect to u in the ball S. The value of the constant $J(u_0)$ is not important for our goals, and so we can set it to zero. Therefore, the formula (3.98) takes the following simple form:

$$J(v) = \int_0^1 \langle A(tv), v \rangle \, dt. \tag{3.99}$$

It is this formula that will be used below.

Summarizing, we have proved that the condition (3.96) implies the equality (3.98). Now we prove that the expression (3.98) is really the unknown potential, i.e., the relation (3.98) implies (3.96). In order to do this, we select some elements u, $u + \varphi$ in the ball S and rewrite the equality (3.98) as follows:

$$\begin{aligned} J(u + \varphi) - J(u) = \int_0^1 & [\langle A(u_0 + t(u + \varphi - u_0)), u + \varphi - u_0 \rangle \\ & - \langle A(u_0 + t(u - u_0)), u - u_0 \rangle \, dt. \end{aligned}$$

We make the following simple transformations:

$$\begin{aligned} J(u + \varphi) - J(u) = & \int_0^1 \langle A(u_0 + t(u - u_0) + t\varphi), \varphi \rangle \, dt \\ & + \int_0^1 \langle A(u_0 + t(u - u_0) + t\varphi) \\ & - A(u_0 + t(u - u_0)), u - u_0 \rangle \, dt. \end{aligned} \tag{3.100}$$

Using the formula

$$\frac{d}{dt} J(u + t(v - u)) = J'(u + t(v - u), v - u), \tag{3.101}$$

we can rewrite the second term in the equality (3.100), denoted by I, as follows:

$$\begin{aligned} I &= \int_0^1 \int_0^t \frac{\partial}{\partial s} \langle A(u_0 + t(u - u_0) + s\varphi), u - u_0 \rangle \, ds \, dt \\ &= \int_0^1 \int_0^t \langle A'(u_0 + t(u - u_0) + s\varphi, \varphi), u - u_0 \rangle \, ds \, dt. \end{aligned} \tag{3.102}$$

Let us apply our basic hypothesis (3.82) in the integrand and change the order of the integration:

$$I = \int_0^1 \int_0^t \langle A'(u_0 + t(u - u_0) + s\varphi, u - u_0), \varphi \rangle \, ds \, dt$$
$$= \int_0^1 \int_s^1 \langle A'(u_0 + t(u - u_0) + s\varphi, u - u_0), \varphi \rangle \, dt \, ds. \tag{3.103}$$

Again, using the equality of the type (3.101), calculate the inner integral

$$I = \int_0^1 \langle A(u_0 + s(u - u_0) + s\varphi) - A(u_0 + s(u - u_0) + s\varphi), \varphi \rangle \, ds. \tag{3.104}$$

Let us replace the variables of integration by t and put the result into the formula (3.100):

$$J(u + \varphi) - J(u) = \int_0^1 [\langle A(u_0 + t(u - u_0) + t\varphi), \varphi \rangle + \langle A(u + t\varphi), \varphi \rangle$$
$$- \langle A(u_0 + t(u - u_0) + t\varphi), \varphi \rangle] \, dt$$
$$= \int_0^1 \langle A(u + t\varphi), \varphi \rangle \, dt. \tag{3.105}$$

Thus, for all φ

$$J(u + s\varphi) - J(u) = \int_0^1 \langle A(u + ts\varphi), s\varphi \rangle \, dt. \tag{3.106}$$

Hence

$$\frac{J(u + s\varphi) - J(u)}{s} = \int_0^1 \langle A(u + ts\varphi), \varphi \rangle \, dt. \tag{3.107}$$

Letting $s \to 0$ shows

$$J'(u, \varphi) = \langle A(u), \varphi \rangle,$$

i.e., $\nabla J = A$ which completes the proof.

We now give an example of checking the conditions (3.86) and the application of the formula (3.99). We turn to the problem of the theory of elasticity (2.217) and (2.218). Operator A of this problem can be defined by the equation (2.131)

$$\langle A(u), v \rangle = \int_\Omega a_{ijkl} \varepsilon_{kl}(u) \varepsilon_{ij}(v) \, d\Omega, \quad \langle \rho F, v \rangle = \int_\Omega \rho F \cdot v \, d\Omega. \tag{3.108}$$

The expression $\langle A'(u, \varphi), \psi \rangle$ on the left-hand side of the condition (3.86) is the following:

$$\langle A'(u, \varphi), \psi \rangle = \int_\Omega a_{ijkl} \varepsilon_{kl}(\varphi) \varepsilon_{ij}(\psi) \, d\Omega. \tag{3.109}$$

The symmetry (2.112) of the elasticity tensor modulus a_{ijkl} with respect to the first and second pairs of the index immediately permits us to conclude that a potential exists because the condition (3.86) is satisfied.

Now calculate the potential of this operator by the formula (3.99):

$$J(v) = \int_0^1 \int_\Omega a_{ijkl}\varepsilon_{kl}(tv)\varepsilon_{ij}(v)\, d\Omega = \frac{1}{2}\int_\Omega a_{ijkl}\varepsilon_{kl}(v)\varepsilon_{ij}(v)\, d\Omega. \qquad (3.110)$$

This expression coincides with the expression (2.224) given before.

Readers might now consider the examples of the full analysis starting with the local (differential) setting of the problem and finishing with the computation of the potential, checking its strict convexity, which provides (together with differentiability and continuity) the existence and uniqueness for the problem of the minimization of potential.

3.5 Boundary value problems in the Hencky–Ilyushin theory of plasticity without discharge

The formulation of the equations of this theory, which was first given in the book by A. A. Ilyushin [Ily49] can be found in many works. In the following we give a short variant of it.

Represent the stress tensor σ_{ij} and the small strain tensor ε_{ij} as the sum of spherical and deviator parts:

$$\sigma_{ij} = \sigma_{ij}^D + \frac{1}{3}(\sigma_{kk})\delta_{ij}, \quad \varepsilon_{ij} = \varepsilon_{ij}^D + \frac{1}{3}(\varepsilon_{kk})\delta_{ij}, \qquad (3.111)$$

where $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, and δ_{ij} is the Kronecker symbol. As it is known, the spherical part $\sigma_{kk}/3 \equiv -p$ is equal to the average pressure at the given point, the quantity $\varepsilon_{kk}/3 \equiv \theta/3$ equals the relative change in the volume of the neighborhood of the given point, and the deviator stress part σ_{ij}^D characterizes the deviation state of the stresses from the pure pressure state. The deviator parts of the strains ε_{ij}^D characterizes the deviation of the deformed state from pure extension or compression, i.e., it describes the change in the form of an infinitely small element.

The basic hypotheses are the following:

1. The relation between the spherical parts is linear:

$$-p = K\theta, \quad K = \text{const}. \qquad (3.112)$$

2. The deviator parts are proportional to one another, and the coefficient of the proportionality depends only on the distance of the points representing the strain or stress state in the Cartesian space of nine dimensions from the origin:

$$\sigma_{ij}^D = k(\varepsilon_{pq}^D\varepsilon_{pq}^D)\varepsilon_{ij}^D. \qquad (3.113)$$

(The repeated index means, as usual, summation from 1 to 3.) The quantity

$$e_u^2 = \frac{2}{3}\varepsilon_{pq}^D \varepsilon_{pq}^D \tag{3.114}$$

is called the *strain intensity*. The quantity

$$\sigma_u^2 = \frac{3}{2}\sigma_{pq}^D \sigma_{pq}^D \tag{3.115}$$

is called the *stress intensity*.

The coefficient k in the formula (3.113) is defined by the unique function which represents the dependence of the stress intensity σ_u on the strain intensity e_u:

$$\sigma_u = \Phi(e_u). \tag{3.116}$$

Indeed, squaring and adding the left- and right-hand sides of the formula (3.113) for all the possible values of the indices i and j, we obtain

$$\sigma_{pq}^D \sigma_{pq}^D = k^2 \varepsilon_{rs}^D \varepsilon_{rs}^D. \tag{3.117}$$

Taking into account the definitions (3.114) and (3.115), we can rewrite the formula (3.117) as follows:

$$\frac{2}{3}\sigma_u^2 = k^2 \frac{3}{2} e_u^2. \tag{3.118}$$

From (3.118) and (3.112) it follows the expression for the coefficient k:

$$k = \frac{2\Phi(e_u)}{3e_u}. \tag{3.119}$$

The dependence $\Phi(e_u)$ can be obtained from an experiment where only shear strains arise (see, e.g., [Ily49]). Substituting (3.119) into (3.113), we obtain

$$\sigma_{ij}^D = \frac{2\Phi(e_u)}{3e_u}\varepsilon_{ij}^D. \tag{3.120}$$

Taking into account the decomposition (3.111) and the linear dependence of spherical parts, we obtain the main governing equation:

$$\sigma_{ij} = \frac{2\Phi(e_u)}{3e_u}\left(\varepsilon_{if} - \frac{1}{3}\theta\delta_{ij}\right) + K\theta\delta_{ij}, \tag{3.121}$$

which defines the nonlinear theory of elasticity of an isotropic body. This relation coincides with the governing equation of the *strain theory of plasticity* where the process is active, i.e., the strain intensity increases for the whole time of loading.

We use a small parameter to estimate the deviation of strain theory of plasticity from the linear elasticity theory (3.121) described by the Hooke equation

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \left(K - \frac{2}{3}\mu\right)\theta\delta_{ij}. \tag{3.122}$$

We introduce now the dimensionless function $\omega(e_u)$:

$$\sigma_u = 3\mu e_u[1 - \omega(e_u)], \tag{3.123}$$

$$\omega(e_u) = 1 - \frac{\Phi(e_u)}{3\mu e_u}, \tag{3.124}$$

where μ denotes the shear modulus of the isotropic body. Now substitute the expression (3.123) into the equation (3.121) and rearrange the terms in order to select the *linearly elastic component* (3.122)

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - 2\mu\omega(e_u)\varepsilon_{ij}^D, \tag{3.125}$$

where $\lambda = K - 2\mu/3$ is the Lamé parameter. Now, putting the expression (3.125) into the equilibrium equation

$$\frac{\partial\sigma_{ij}}{\partial x_j} + \rho F_i = 0, \tag{3.126}$$

we obtain the differential equation with respect to the displacements:

$$(\lambda + \mu)\frac{\partial}{\partial x_i}(\operatorname{div} u) + \mu\Delta u_i - 2\mu\frac{\partial}{\partial x_j}[\omega(e_u(u))\varepsilon_{ij}^D(u)] + \rho F_i = 0, \tag{3.127}$$

which should be completed with the appropriate boundary conditions, e.g.,

$$u|_\Sigma = 0. \tag{3.128}$$

To transform this boundary value problem to the variational one, we define functional space V of the solutions as $V = [H_0^1(\Omega)]^n \equiv H_0^1(\Omega)$. Note that such a choice is valid under additional constraints for function ω, see the conditions (3.136). Repeating the reasoning used in the construction of the variational equation (2.220), we arrive at the equation

$$\int_\Omega [\lambda\theta(u)\theta(v) + 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v)]\,d\Omega - 2\mu\int_\Omega \omega(e_u)\varepsilon_{ij}^D(u)\varepsilon_{ij}^D(v)\,d\Omega$$
$$= \int_\Omega \rho F\cdot v\,d\Omega \equiv L(v) \quad \forall v \in V. \tag{3.129}$$

Any solution of the equation (3.127) satisfies the variational equation (3.129). Any solution of the equation (3.129), having second derivatives, satisfies the equation (3.127). So, with this restriction the equations (3.127) and (3.129) are equivalent. The solutions of the equation (3.129) are called generalized solutions of the equation (3.127).

Let us define the operator A of the problem as follows:

$$\langle A(u), v\rangle = \int_{\Omega} [\lambda\theta(u)\theta(v) + 2\mu\varepsilon_{ij}(u)\varepsilon_{ij}(v) - 2\mu\omega(e_u(u))\varepsilon^D_{ij}(u)\varepsilon^D_{ij}(v)]\, d\Omega \quad \forall v \in V. \tag{3.130}$$

We prove that this operator has a potential and we calculate it. In order to prove the potentiality, we use the criterion (3.126), under the assumption that the function $\omega(e_u)$ is differentiable and omitting the proof of continuity of the derivative $A'(u,\varphi)$ with respect to u and φ. We have

$$\langle A'(u,\varphi), \psi\rangle = \int_{\Omega} \Big\{ \lambda\theta(\varphi)\theta(\psi) + 2\mu\varepsilon_{ij}(\varphi)\varepsilon_{ij}(\psi) - 2\mu \Big[\omega(e_u(u))e_u(\varphi)e_u(\psi) + \frac{2}{3}\frac{d\omega(e_u(u))}{de_u}\frac{\varepsilon^D_{pq}(u)\varepsilon^D_{pq}(\varphi)}{e_u(u)}\varepsilon^D_{ij}(u)\varepsilon^D_{ij}(\psi) \Big] \Big\} d\Omega. \tag{3.131}$$

From this expression it follows that

$$\langle A'(u,\varphi), \psi\rangle = \langle A'(u,\psi), \varphi\rangle \quad \forall u, \varphi, \psi \in V,$$

which means that A is a potential operator with the potential

$$J(v) = \int_0^1 \langle A(tv), v\rangle\, dt = \frac{1}{2}\int_{\Omega} [\lambda\theta^2(v) + \mu\varepsilon_{ij}(v)\varepsilon_{ij}(v)]\, d\Omega - 3\mu\int_{\Omega}\int_0^{e_u(v)} \omega(s) s ds d\Omega \equiv J_0(v) - j(v), \tag{3.132}$$

where, for convenience, the non-quadratical part $j(v)$ of the functional $J(v)$ has been separated out:

$$j(v) = 3\mu\int_{\Omega}\int_0^{e_u(v)} \omega(s) s\, ds\, d\Omega. \tag{3.133}$$

The variational equation (3.129) and the minimization problem for the functional $\Pi(v) \equiv J(v) - L(v)$ on the space V are equivalent. We prove that the minimization problem has a unique solution. In order to do this, according to Theorems 3.8 and 3.9, we have to prove that

$$\lim_{\|v\|\to\infty} J(v) = +\infty, \tag{3.134}$$

$$J''(u,\varphi,\varphi) > 0 \quad \forall u, \varphi \in V,\ \varphi \neq 0. \tag{3.135}$$

Assume that the graph of the function $\sigma_u = \Phi(e_u)$ has the form shown in Figure 3.1. Since $\Phi(e_u)$ is convex upwards, it is monotonically increasing and its nonlinear part starts from some point (e_s, σ_s). These properties are equivalent to the inequalities

$$\omega(s) \leq \omega(s) + \omega'(s)s < 1, \quad 0 < \omega(s) \tag{3.136}$$

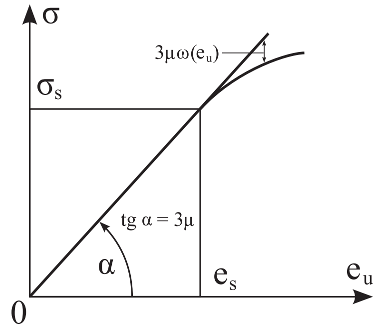

Fig. 3.1. Graph of the stress–strain relation

which are true in the domain of nonlinearity. (In the domain of linearity $\omega = 0$ and the problem is reduced to the well-known theorems of the linear theory of elasticity.)

We prove the *coercivity* property (3.134) of the functional (3.132). First, we observe that the Korn inequality (see [LVG02, p. 86]) and the condition $\|v\|_V \to +\infty$ imply

$$\int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v)\, d\Omega \longrightarrow +\infty, \quad \text{as } \|v\| \to +\infty, \tag{3.137}$$

i.e., either $\theta^2(v) \to +\infty$ or $e_u(v) \to +\infty$ or $\theta^2(v) \to +\infty$, $e_u(v) \to +\infty$. Clearly, in the first case the condition (3.134) is satisfied. Therefore, we have to analyze the case where $e_u(v) \to +\infty$ as $\|v\|_V \to +\infty$.

Choose a number $\varepsilon > 0$ and find $\delta = \delta(\varepsilon)$ with the property

$$3\mu \int_0^{e_s+\delta} \omega(s)sds = 3\mu \int_{e_s}^{e_s+\delta} \omega(s)sds < 3\mu\varepsilon. \tag{3.138}$$

For the case $e_{u(v)} \geq e_s + \delta$ we have the strict inequality

$$\omega(s)s < (1-\alpha)s, \quad 0 < \alpha = \text{const} < 1. \tag{3.139}$$

Therefore,

$$\begin{aligned} J(v) = &\int_\Omega \Bigg[\frac{\lambda}{2}\theta^2(v) + \mu\varepsilon_{ij}(v)\varepsilon_{ij}(v) - 3\mu \int_{e_s}^{e_s+\delta} \omega(s)sds \\ &- 3\mu \int_{e_s+\delta}^{e_u(v)} \omega(s)sds\Bigg] d\Omega \\ &> \int_\Omega \Bigg[\frac{\lambda}{2}\theta^2(v) + \mu\varepsilon_{ij}(v)\varepsilon_{ij}(v) \\ &- 3\mu\varepsilon - \frac{3}{2}\mu(1-\alpha)e_u^2(v) + \frac{3}{2}\mu(1-\alpha)(e_s+\delta)^2\Bigg] d\Omega. \end{aligned} \tag{3.140}$$

Noting that, according to the definitions (3.111) and (3.114),

$$\varepsilon_{ij}(v)\varepsilon_{ij}(v) = \frac{3}{2}e_u^2(v) + \frac{1}{3}\theta^2(v), \tag{3.141}$$

from the inequality (3.140) we find

$$J(v) > \int_\Omega \left[\left(\frac{\lambda}{2} + \frac{\mu}{3} \right) \theta^2(v) + \frac{3}{2}\alpha\mu e_u^2(v) + \frac{3}{2}\mu(1-\alpha)(e_s+\delta)^2 \right] d\Omega \to \infty$$

when $\|v\|_V^2 \to \infty$.

Notice that the part

$$\int_\Omega \rho F \cdot v \, d\Omega$$

of the operator in the equation (3.129) does not affect the property (3.134) in the proof, because under the assumptions made for smoothness, the estimate

$$\left| \int_\Omega \rho F \cdot v \, d\Omega \right| \leq \|\rho F\|_V^* \|v\|_V \tag{3.142}$$

holds and when $\|v\|_V \to +\infty$ the square of the norm v dominates this.

We now prove the inequality (3.135). Twice differentiation of the functional (3.132) gives the formula

$$J''(u,\varphi,\varphi) = \int_\Omega \Big\{ \lambda\theta^2(\varphi) + 2\mu\varepsilon_{ij}(\varphi)\varepsilon_{ij}(\varphi) - 2\mu\left[\omega(e_u(u))\frac{3}{2}e_u^2(\varphi) + \omega'(e_u(u))\frac{2}{3e_u(u)}(\varepsilon_{ij}^D(u)\varepsilon_{ij}^D(\varphi))^2 \right] \Big\} d\Omega. \tag{3.143}$$

Applying the Cauchy inequality to the sum

$$\left[\varepsilon_{ij}^D(u)\varepsilon_{ij}^D(\varphi)\right]^2 \leq \frac{9}{4}e_u^2(u)e_u^2(\varphi), \tag{3.144}$$

from the formula (3.143) we obtain the inequality

$$J''(u,\varphi,\varphi) \leq \int_\Omega \{ \lambda\theta^2(\varphi) + 2\mu\varepsilon_{ij}(\varphi)\varepsilon_{ij}(\varphi) - 2\mu\left[\omega(e_u(u)) + \omega'(e_u(u))e_u(u)\right] \} e_u^2(\varphi)\, d\Omega, \tag{3.145}$$

which, together with the Korn inequality and the assumption (3.136), implies the condition (3.135).

So, we proved that the boundary value problems (3.127) and (3.128) of the Hencky–Ilyushin theory of plasticity without discharge is equivalent to the variational equation (3.129), and the problem of the minimization of the functional

$$\Pi(v) = J_0(v) - j(v) - L(v) \tag{3.146}$$

in the space V has a solution, being at least a generalized one, and this solution is unique.

Problems with other kinds of boundary conditions (like the conditions (2.116) and (2.117)) can be investigated in the same manner. In these problems there are some specific questions (e.g., the problem of the smoothness of the solution at a point of change in the boundary conditions, the problem of choosing, and the smoothness of the particular solution chosen, required to pass from the nonhomogeneous to the homogeneous boundary condition). These questions will be not considered because they are only marginally relevant to the variational investigation of BVPs.

Notice in conclusion of this section that another approach to the elastic-plastic behavior of industrial material and the corresponding variational theory is developed in [NH80].

3.6 Problems in the elastic bodies theory with finite displacements and strain

The geometrically nonlinear theory of the elasticity is one of the most complicated in the theory of the mechanics of solids. The base of this theory can be found in the literature (e.g., in [Ogd84, GA60]). In the following we consider the variational aspects of the solutions of some typical classes of the geometrically nonlinear theory. First of all, we give some basic definitions.

In the formulation of geometrically nonlinear problems two different approaches are used, namely, the Lagrange and Euler approaches. In the Lagrange method the coordinates of the investigated domain at some fixed position are chosen as the independent variables. The basic unknown kinematic variables describe the current position of the particles. In the Euler method, which is used primarily in the mechanics of fluids and gases, the independent variables are the coordinates of the space points where the particles of the investigated domain are moving. The basic unknown kinematic variables are the velocities of the particles in that space and the velocity field.

The mechanics of solids deals with bounded domains and has two choices respecting the independent variables. In the first variant the independent variables are the coordinates of the points of the domain in its initial position, particularly for a state without stress or strain. In the second variant, the independent variables run over the domain occupied by the solid after deformation. The advantage of the first method is that the domain definition of the independent variables is known. The disadvantage is that the external actions are operators of the unknown solution, obtained by the recalculation of the current values of the external actions with respect to the initial position of the solid. The second method is preferable when the deformed state of the body is known (at least, approximately). Here, it is simpler to operate on the given external actions.

In both versions it is necessary to define the connection between certain sets of vectors in the initial and deformed states of the body. Therefore, the use of systems of curvilinear coordinates is a necessary element of the geometrically nonlinear theory. We consider "natural" systems of curvilinear coordinates, defined by the deformation process.

3.6.1 Vectors and strains in systems of curvilinear coordinates

Let the deformation process consist of the transition from its initial state in the domain Ω_0 with the boundary Σ_0 to the domain Ω with boundary Σ as a result of external actions. The coordinates of the particles of the domains Ω_0 and Ω are given with respect to some global Cartesian system of coordinate with unit basis vectors k_i, $i = 1, 2, 3$. We have (see the formula (1.18)):

$$a \in \Omega_0, \quad a = a^i k_i, \quad x \in \Omega, \quad x = x^i k_i. \tag{3.147}$$

The change of the position of the particles is given by the formula (2.82):

$$x^i = x^i(a^1, a^2, a^3). \tag{3.148}$$

Recall that in Chapter 1 this formula was interpreted as the introduction of a system of curvilinear coordinates in the domain Ω: the coordinate line $a^2 = \text{const}$, $a^3 = \text{const}$, $a^1 \neq \text{const}$, being in the domain Ω a straight line which is parallel to the vector k_1. As a result of the transformation (3.148), this line is transformed to some curved line in the domain Ω. The collection of three such coordinate lines in the domain Ω, existing at every point due to the definition, is the system of curvilinear coordinates in this domain. For this curvilinear coordinate system at every point of the domain Ω we defined the oblique $\{G_i\}_{i=1}^{3}$, where the vector G_i is tangent to the coordinate lines (Figure 3.2):

$$G_1 = \frac{\partial x}{\partial a^1}, \quad G_2 = \frac{\partial x}{\partial a^2}, \quad G_3 = \frac{\partial x}{\partial a^3}. \tag{3.149}$$

In the computations and the reasoning we use the representation of the vectors and strains in a system of the basis vectors $\{G_1, G_2, G_3\}$, which are,

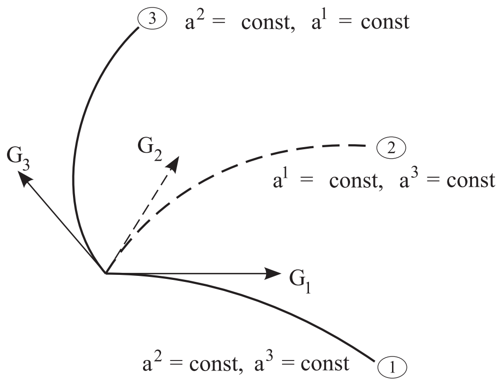

Fig. 3.2. Coordinate lines and local basis

in general, neither unit nor orthogonal. We summarized the basic formulae and definitions in Chapter 1.

We now consider a field of the vector Q, with the contravariant components Q^i being the projection onto the vector G_i, and covariant component G_i defined as the projection of the vector Q onto the basis vector of the conjugate basis. This basis is defined using the contravariant components G^{ij} of the metric tensor $\hat{G}$, by the formula

$$G^i = G^{ij} G_j \equiv G^{i1} G_1 + G^{i2} G_2 + G^{i3} G_3, \quad i = 1, 2, 3, \tag{3.150}$$

where G^{ij} is the solution of the linear systems of the equations

$$G^{ik} G_{kj} = \delta^i_j. \tag{3.151}$$

Here δ^i_j is the Kronecker symbol.

Using the diadic multiplication (see (1.8)), we obtain the representation of the metric tensor $\hat{G}$:

$$\hat{G} = G^{ij} G_i \otimes G_j = G_{ij} G^i \otimes G^j = \delta^i_j G_i \otimes G^j. \tag{3.152}$$

For an arbitrary tensor $\hat{T}$ of the second order we use the representations

$$\hat{T} = T^{ij} G_i \otimes G_j = T_{ij} G^i \otimes G^j = T^{i\cdot}_{\cdot j} G_i \otimes G^j = T^{\cdot j}_{i \cdot} G^i \otimes G_j, \tag{3.153}$$

where T^{ij}, T_{ij}, $T^{i\cdot}_{\cdot j}$, $T^{\cdot j}_{i\cdot}$ are the coefficients of the tensor $\hat{T}$ decomposition depending on the tensor basis $G_i \otimes G_j, \ldots$.

We now deduce in more detail the covariant derivative of the contravariant and covariant component of the field Q introduced in Chapter 1. Find the total variation dQ of the field Q, while moving from the given point $x = x(a)$ to the infinitesimally close point $x = x(a + da)$ through the components da^i of the infinitesimally small vector da. We have

$$dQ = d(Q^p G_p) = \frac{\partial}{\partial a^i}(Q^p G_p) da^i = \left(\frac{\partial Q^p}{\partial a^i} G_p + Q^p \frac{\partial G_p}{\partial a^i} \right) da^i. \tag{3.154}$$

Let us introduce the decomposition

$$\frac{\partial G_p}{\partial a^i} = \Gamma^k_{pi} G_k. \tag{3.155}$$

Recall that the set of coefficients Γ^k_{pi}, having the property of symmetry $\Gamma^k_{pi} = \Gamma^k_{ip}$ (which follows from the definition (3.149)), are the *Christoffel symbol of the second kind.*

Substitution of the decomposition (3.155) into (3.154) leads to

$$dQ = \left(\frac{\partial Q^k}{\partial a^i} G_k + Q^p \Gamma^k_{pi} G_k \right) da^i \left(\frac{\partial Q^k}{\partial a^i} + Q^p \Gamma^k_{pi} \right) G_k da^i \equiv \nabla_i Q^k G_k da^i. \tag{3.156}$$

The formula (3.156) gives the variation dQ. The quantity

$$\nabla_i Q^k = \frac{\partial Q^k}{\partial a^i} + Q^p \Gamma^k_{pi} \tag{3.157}$$

is called the *covariant derivative of the contravariant components* Q^i of the vector Q.

Using the formula $G^i \cdot G_j = \delta^i_j$ and the equality (which is implied by it)

$$\frac{\partial G^i}{\partial a^k} \cdot G_j = -G^i \cdot \frac{\partial G_j}{\partial a^k} = -\Gamma^i_{jk}, \tag{3.158}$$

we can find the expression for the *covariant derivative of the covariant components*

$$\nabla_i Q_j = \frac{\partial Q_j}{\partial a^i} - Q_k \Gamma^k_{ij}. \tag{3.159}$$

Using the formulae (3.155) and (3.156), we can calculate the covariant derivative of any component of the second-order tensor.

There exists another definition of the Lagrange coordinates [GA60] where the coordinates of the particles before and after deformation or any appropriate domain of the independent variables play the same role. For this, introduce a general system of curvilinear coordinates with the coordinates $\theta^i \equiv \theta_i$, in which

$$a^i = a^i(\theta^1, \theta^2, \theta^3), \quad x^i = x^i(\theta^1, \theta^2, \theta^3). \tag{3.160}$$

Note that, in general, the coordinates x^i depend on the time. Then, instead of the definition of the basis vectors G_i of the "natural" curvilinear system (a_1, a_2, a_3), we use the definition

$$G_i = \frac{\partial x}{\partial \theta^i} \equiv x_{,i}. \tag{3.161}$$

We give the definition of the basis vectors $g_i = da/d\theta^i \equiv a_{,i}$ of the curvilinear system θ^i in the domain Ω. All formulae given before (and later) where the vectors G_i appears remain valid. Moreover, we can write the corresponding analogies for the vectors g_i, e.g.,

- Definition of the metric tensor components in the initial (nondeformed) state Ω_0
$$g_{ij} = g_i \cdot g_j, \quad \|g^{ij}\| = \|q_{ij}\|^{-1}; \tag{3.162}$$
- Formula for the Christoffel symbol for this state

$$\overset{\circ}{\Gamma}{}^k_{pi} = \frac{\partial g_p}{\partial \theta^i} \cdot g^k \equiv g^k_{g,i} \cdot g^k; \tag{3.163}$$

- Definition of the Hamilton operator

$$\overset{\circ}{\nabla} = g^p \overset{\circ}{\nabla}_p, \tag{3.164}$$

whose components $\overset{\circ}{\nabla}_p$ are defined by the formulae (3.157) and (3.159) by changing the variables a^i to θ^i and the Christoffel symbol Γ^k_{ij} to $\overset{\circ}{\Gamma}{}^k_{ij}$.

The system of variables θ^i is also called a Lagrange-type system. Its main advantage is its easy conversion to any other system. For example, by putting $\theta^i = a^i$, we return to the already examined "natural" curvilinear coordinate system, because in this case $\theta_i = k_i$. For practical calculations the case $\theta^i = x^i$ is important. Here

$$G_i = k_i, \quad g_i = \frac{\partial a}{\partial x^i},$$

i.e., the current coordinate system is the Cartesian one, the corresponding coordinate system is curvilinear and it is generated by the set of straight lines in the deformed state.

Later we mainly use the variant $\theta^i = a^i$. However, we will take into account that this particular choice can easily be transformed to the general case of Lagrange coordinates by changing a^i to θ^i, by substituting the partial differentiation by the covariant one with metric tensor θ_{ij}, etc.

3.6.2 Strains and stress

In order to describe finite strains in Lagrange coordinates, we use the Green strain tensor which was defined in (2.92) as a particular case. The invariant definition of the tensor is the following:

$$\hat{\varepsilon}^G = \frac{1}{2}(\hat{G} - \hat{g}), \tag{3.165}$$

where $\hat{G}$ is the metric tensor of the deformed state and $\hat{g}$ is that of the initial state. If $\theta^i = a^i$, we have

$$\varepsilon^G_{ij} = \frac{1}{2}(G_{ij} - \delta_{ij}). \tag{3.166}$$

We introduce now the displacement vector u by the formula $x = a + u$. Using the decomposition of the form $u = u^i k_i = U^i G_i = U_i G^i$, we obtain

$$\varepsilon^G_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial a^j} + \frac{\partial u^j}{\partial a^i} + \frac{\partial u^m}{\partial a^i}\frac{\partial u_m}{\partial a^j}\right) = \frac{1}{2}\left(\nabla_i U_j + \nabla_j U_i - \nabla_i U^m \nabla_j U_m\right). \tag{3.167}$$

Notice that in a curvilinear Lagrange coordinate system instead of relation (3.167) we obtain:

$$\varepsilon_{ij}^{G} = \frac{1}{2}\left(\overset{\circ}{\nabla}_j u_i + \overset{\circ}{\nabla}_i u_j + \overset{\circ}{\nabla}_i u^m \overset{\circ}{\nabla}_j u_m\right).$$

The strain is the relative variation in the infinitely small volume at the given point. For its computation, choose as a representative the volume dV_0, in Ω_0, a cube with the sides parallel to the global Cartesian coordinate system and the length of the sides equal to da^1, da^2, da^3, respectively. As the result of the deformation, the cube is transformed into an infinitely small oblique parallelepiped with volume dV, the sides of which are defined by the vectors $G_1 da^1$, $G_2 da^2$, $G_3 da^3$, respectively. We have

$$dV_0 = da^1 da^2 da^3, \quad dV = G_1 \cdot (G_2 \times G_3) da^1 da^2 da^3. \tag{3.168}$$

(We assume that the mixed product $G_1 \cdot (G_2 \times G_3)$ is positive. Recall that "$\times$" denote the vector product, "$\cdot$" denote the scalar product.) Introduce the Cartesian components G_i^s of the basis vectors G_i (which differs from the mixed components $G_i^j \equiv \delta_i^j$ of the metric tensor) by the formula

$$G_i = G_i^s k_s, \tag{3.169}$$

where $\{k_s\}_{s=1}^3$ are the unite vectors of a Cartesian basis. We assume that the orientation of the local basis is not changed and coincides with the orientation of the initial basis in the deformation process, then

$$dV = \varepsilon_{ijk} G_1^i G_2^j G_3^j dV_0 = \det \|G_j^i\| dV_0, \tag{3.170}$$

where $\|G_j^i\|$ is a matrix with columns G_1, G_2, G_3. The last equality (3.170) follows from the property of the Levi-Civita symbols ε_{ijk}. Hence, the relative variation of the volume is

$$dV/dV_0 = \det \|G_j^i\|.$$

On the other hand,

$$\det \|G_{ij}\| \equiv G = (\det \|G_j^i\|)^2.$$

The proof of this equality is the following. From the properties of the Levi-Civita symbols and the relation $G_{ij} = G_i^s G_j^s$ we have

$$\varepsilon_{pqr}\varepsilon_{pqr} \det \|G_{ij}\| = \varepsilon_{ijk}\varepsilon_{pkr} G_i^s G_p^s G G_q^t G_k^u G_r^u = \varepsilon_{stu} \det \|G_i^j\| \varepsilon_{stu} \det \|G_i^j\|.$$

The final result is the following:

$$\frac{\partial V}{\partial V_0} = \det \|G_i^j\| = \sqrt{\det \|G_{ij}\|} = \sqrt{G}. \tag{3.171}$$

The quantity $\det \|G_i^j\|$ is the invariant (usually called the *third invariant* [Sok64]) and it is independent on the choice of the coordinate system. This follows from the fact that the obtained result is independent of the choice of the form of the infinitesimally small volume dV_0.

The last set of kinematic formulae, required for the formulation and the solution of these problems, connects the area of the elementary (infinitesimally small) area before and after deformation.

Define the orientated area of the infinitesimally small side of the coordinate parallelepiped (oblique-angled, in general) by the formula

$$dS^r = dS_r G^r = dx^j \times dx^k = G_j \times G_k da^j da^k, \qquad (3.172)$$

where $r \neq j \neq k$, and the vectors $\{G_j, G_k, G_r\}$ form a basis with the orientation coinciding with the original one. Denote by dS_0^r the area of the coordinate parallelepiped before deformation. It follows from the formula (3.172) that

$$dS^r = \sqrt{G} G^{\underline{r}} dS_0^{\underline{r}} = \sqrt{G G^{\underline{rr}}} dS_0^{\underline{r}}, \qquad (3.173)$$

where

$$G^{\underline{rr}} = G^{\underline{r}} \cdot G^{\underline{r}}.$$

(Recall that the underlined index means the absence of the summation.)

We obtain two formulae (for the proof, see, e.g., [Ogd84]) which will be used later:

1. The connection between the area $dS^{(\nu)}$ of the coordinate tetrahedron oblique side and the area dS^r of the sides orthogonal to the coordinate axis (see Figure 3.3)
$$dS^r = \sqrt{G^{\underline{rr}}} dS^{(\nu)} \nu_{\underline{r}}, \qquad (3.174)$$
where $\nu_r = \nu \cdot G_r$, ν is the unique outward drawn normal vector to the surface $dS^{(\nu)}$.

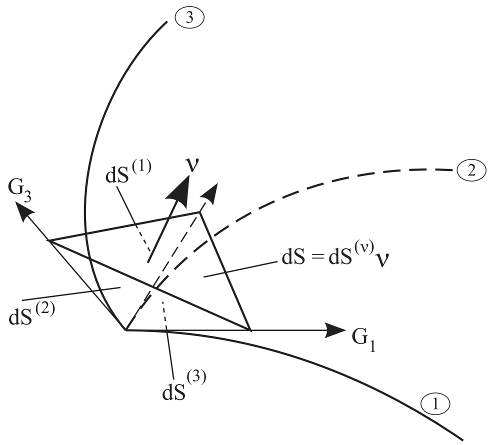

Fig. 3.3. Areas of the sides of infinitesimal tetrahedron

2. The connection of the area dS of any area element with the area dS_0 of this element before deformation is

$$k = k(u) = \frac{dS}{dS_0} = \sqrt{GG^{rs}}\nu_{0r}\nu_{0s}, \tag{3.175}$$

where ν_0 is the normal to the considered area element before deformation,

$$\nu = \nu_{0r}G^r / \sqrt{G^{rs}}\nu_{0r}\nu_{0s}. \tag{3.176}$$

We also have the formula

$$G^{ij} = \frac{1}{6G}E^{ikt}E^{jrs}G_{kr}G_{ts}. \tag{3.177}$$

We now turn to the stress definitions. As a starting point, we use the term of the density vector of the surface tractions $t^{(\nu)}$ on the small area with the normal ν in the strained body, which is described in Section 2.3.2. From the formula for the equilibrium of the infinitesimally small tetrahedron (see Figure 3.3) we have

$$t^{(\nu)}dS^{(\nu)} = \sum_{i=1}^{3} t^i dS_i. \tag{3.178}$$

This relation implies, with the connection (3.174), the equation

$$t^{(\nu)} = \sum_{i=1}^{3} t_i \nu_i \sqrt{G^{\underline{ii}}}, \tag{3.179}$$

which generalizes the equation (2.101). In this equation t_i is the density vector of the surface tractions on the elementary area dS^i (see Figure 3.3).

We now turn from the current curvilinear coordinate system to some other system. The set of numbers $\{\nu_i\}$ is transformed according to the rule of the transformation of the covariant components of the vector. The vector $t^{(\nu)}$ does not change with this transformation. Therefore, the collection of the vectors $t_i\sqrt{G^{\underline{ii}}}$ will be transformed by the transformation rule of the contravariant components for each value of the index i. Then for all i we can introduce a vector with the components τ by the formula

$$t_i\sqrt{G^{\underline{ii}}} = \tau^{ij}G_j. \tag{3.180}$$

The set of numbers $\{\tau^{ij}\}$ is the collection of the contravariant components of a tensor which is called the *stress tensor*. The invariant definition of this tensor, which is denoted by $\hat{t}$, follows from the formula (3.179) and the definition (3.180):

$$t^{(\nu)} = \nu \cdot \hat{t}. \tag{3.181}$$

In the coordinate system $\{G_1, G_2, G_3\}$ we have

$$\hat{t} = \tau^{ij}G_i \otimes G_j = \tau_{ij}G^i \otimes G^j = \dots. \tag{3.182}$$

In the fixed Cartesian coordinate system (with basis k_i) we have

$$\hat{t} = t_-^{ij} k_i \otimes k_i = \pi^{ij} G_i \otimes k_j, \tag{3.183}$$

where t_-^{ij} are the components which are used in the Euler method of the finite strain analysis.

Tensor $\hat{t}$, defined by the density of surface tractions with respect to the unique area of the deformed body (according to the formula (3.158)), is also called the *real stress tensor*, which is different from the *conditional stress tensor* $\hat{t}_0$. The conditional stress tensor is calculated per unit surface area element in the initial state.

To define the tensor $\hat{t}_0$, write the definition of the conditional stress vector $\hat{t}_0^{(\nu)}$ as follows:

$$t_0^{(\nu)} dS_0(\nu) = t^{(\nu)} dS(\nu), \tag{3.184}$$

where the elementary areas quantities $dS^{(\nu)}$ and $dS_0^{(\nu)}$ are connected via the relation (3.175). Introduce further the vector $\hat{t}_0$ by the formula

$$t_0^{(\nu)} = \nu_0 \cdot \hat{t}_0, \tag{3.185}$$

analogous to (3.181).

For the component-wise writing use a different coordinate system, e.g.,

$$\hat{t}_0 = t_0^{ij} k_i \otimes k_j. \tag{3.186}$$

In practice, the following representation is widely used:

$$\hat{t}_0 = s_0^{ij} k_i \otimes G_j, \tag{3.187}$$

which, together with (3.185), implies

$$t_0^{(\nu)} = s_0^{ij} \nu_{0i} G_j, \quad \nu_{0i} = \nu_0 \cdot k_i. \tag{3.188}$$

The decomposition (3.188) is sometimes used for the definition of the tensor $\hat{t}_0$ [Ogd84].

Connections between the components τ^{ij}, t_-^{ij}, π^{ij}, t_0^{ij}, s_0^{ij} and others, which arise by the new representation of the stress tensor (being different from (3.182), (3.183), (3.186), and (3.187)), are defined on the basis of the definition (3.149) of the vectors G_i. For example, from the inequalities

$$s_0^{ij} k_i \otimes G_j = s_0^{ij} k_i \otimes \left(\frac{\partial x^p}{\partial a^j} k_p \right) = s_0^{ij} \frac{\partial x^p}{\partial a^j} k_i \otimes k_p = t_0^{ip} k_i \otimes k_p \tag{3.189}$$

it follows the formula

$$t_0^{ip} = s_0^{ij} \frac{\partial x^p}{\partial a^j} = s_0^{ij} \left(\delta_j^p + \frac{\partial u^p}{\partial a^j} \right). \tag{3.190}$$

3.6.3 Equilibrium (motion) equations

First of all, consider the equilibrium and motion equations for the components t_-^{ij} of the stress tensor in the Euler coordinate system. They coincide with the equations (2.103) or (2.104) (with the notations introduced in the current section):

$$\frac{\partial t_-^{ij}}{\partial x^i} + \rho F^j = \rho \frac{dv^j}{dt}, \tag{3.191}$$

where ρ is the current density of the material, $F = F^j k_j$ is the given density of the body forces (in the general case being functions of the unknown solution) $x = x(a)$, and $v^j = du^j/dt$ is the velocity of the particle. As in Section 2.3.2, the symmetry of the components $t^{ij} = t^{ji}$ can be established. Comparing with the representation (3.183) and using the relation

$$\hat{t} = \tau^{ij} G_i \otimes G_j = \tau^{ij} \frac{\partial x^p}{\partial a^i} \frac{\partial x^q}{\partial a^i} k_p \otimes k_q, \tag{3.192}$$

we establish the following symmetry: $\tau^{ij} = \tau^{ji}$.

Consider the motion equation with respect to the variables a^i which are important for the further discussion. Using the change (2.82) of the independent variables in the equation (2.102), we obtain the equation:

$$\int_{\Omega_0} \rho_0 \ddot{u}\, d\Omega_0 = \int_{\Omega_0} \rho_0 F\, d\Omega_0 + \int_{\Sigma_0} t_0^{(\nu)}\, d\Sigma_0, \tag{3.193}$$

where Ω_0 is an arbitrary subdomain of the considered domain. Substituting the expression (3.185) into (3.193), applying the Gauss–Ostrogradski formula and using the arbitrariness of the domain Ω_0, we obtain the equation

$$\frac{\partial t_0^{ij}}{\partial a^i} + \rho_0 F^j = \rho_0 \frac{\partial^2 u^j}{\partial t^2} \tag{3.194}$$

or, with respect to the components s_0^{ij},

$$\frac{\partial}{\partial a^i}\left[s_0^{ir}(\delta_r^j + u_{,r}^j)\right] + \rho_0 F^j = \rho_0 \frac{\partial^2 u^j}{\partial t^2}. \tag{3.195}$$

The comma together with index "r" means the derivative with respect to the variable a^r. Similarly, we can construct the motion equations for other components of the stress tensor as well.

We now obtain the motion equations with respect to the components τ^{ij} in the curvilinear system with the basis vectors G_i. These equations are useful in applications and can be obtained from the equation (2.102) using the Gauss–Ostrogradski theorem in the curvilinear coordinate system

$$\nabla_i \tau^{ij} + \rho F^j = \rho \frac{dv^j}{dt}. \tag{3.196}$$

From the formula

$$\pi^{ij} = \tau^{ir}(\delta_r^j + u_{,r}^j), \quad \tau^{ij} = \pi^{ir}(\delta_r^j - \nabla_r u^j),$$

which follows from the definitions (3.182) and (3.183) and from the equations (3.196), we have the motion equations with respect to the components π^{ij}:

$$\nabla_i[\pi^{ir}(\delta_r^j - \nabla_r u^j)] + \rho F^j = \rho \frac{dv^j}{dt}. \tag{3.196'}$$

We now formulate the boundary condition for the given surface tractions for different forms of the motion equation. For the equation (3.191) we have

$$t^{ij}\nu_i|_\Sigma = P^j, \tag{3.197}$$

where ν_i are Cartesian coordinates in the system $\{k_i\}$ of the unite vector components ν orthogonal to the surface Σ of the body Ω, P^j are the Cartesian vector components of the prescribed efforts P. As already mentioned, the use of the condition (3.197) is difficult, because the position of the surface Σ and the normal vector ν are unknown. Similar problems arise from use of the boundary conditions for the components τ^{ij},

$$\tau^{ij}\nu_i G_j|_\Sigma = P \tag{3.198}$$

(where ν_i are the covariant components of the normal vector ν with respect to the system $\{G_i\}$), and for the components π^{ij},

$$\pi^{ir}(\delta_r^j - \nabla_r u^j)\nu_i g^j|_\Sigma = P. \tag{3.198'}$$

Using the definitions (3.184) and (3.185) and the formula connecting elementary areas before and after deformation, we can find the boundary conditions for the components t_0^{ij} and s_0^{ij} as follows:

$$t_0^{ij}\nu_{0i}|_{\Sigma_0} = P^j(x(a))\sqrt{GG^{rs}}\nu_{0r}\nu_{0s}, \tag{3.199}$$

$$s_0^{ir}(\delta_r^j + u_{,r}^j)\nu_{0i}|_{\Sigma_0} = P^j(x(a))\sqrt{GG^{rs}}\nu_{0r}\nu_{0s}, \quad G = \det\|G_{ij}\|. \tag{3.200}$$

Note that we know the functions $P^j(x(a))$ only but not its argument. This fact imply some difficulties in the solution of the corresponding boundary value problem.

3.6.4 Governing equations

Consider a special class of materials for which the stress tensor is equal to the derivative of the function (called the *elasticity potential*) with respect to the strain tensor. Such materials are called *elastic materials*. Denote the elasticity potential by W and suppose that it depends either on the strain tensor $\hat{\varepsilon}^G$, or on the gradient $\nabla \otimes u$.

In order to define the stress through W we use the first law of thermodynamics in its simplest version: the change in the total energy of an arbitrary subdomain Ω with boundary Σ is equal to the work of the forces acting on this system:

- Surface forces with density $t^{(\nu)}$,
- Volume forces with the density $\rho(F - dv/dt)$, where v is the velocity vector.

In such a formulation we assume that the state parameters of the system are the kinematic variables, either $\hat{\varepsilon}^G$ or $\nabla \otimes u$, variations of which define the change W. We neglect nonmechanical effects like the influence of temperature. Calculate the density W per unit volume of the nondeformed body. Then the mathematical formulation of the first law for the infinitely small variation in the state parameters can be written as follows:

$$\int_\Omega \frac{\rho}{\rho_0} \delta W \, d\Omega = \int_\Sigma t^{(\nu)} \cdot \delta u \, d\Sigma + \int_\Omega \rho \left(F - \frac{dv}{dt} \right) \cdot \delta u \, d\Omega, \tag{3.201}$$

where δu is an infinitesimally small variation which is compatible with the kinematical constraints on the system. We preserve the other notations introduced earlier.

In order to transform the surface integral we use the definition (3.181), the representation (3.182) and the Gauss–Ostrogradski formula in the curvilinear coordinate system $x = x(a)$

$$\begin{aligned} \int_\Sigma t^{(\nu)} \cdot \delta u \, d\Sigma &= \int_\Sigma \nu_r \tau^{rs} \delta u_s \, d\Sigma = \int_\Omega \nabla_r (\tau^{rs} \delta u_s) \, d\Omega \\ &= \int_\Omega (\nabla_r \tau^{rs}) \delta u_s \, d\Omega + \int_\Omega \tau^{rs} \nabla_r \delta u_s \, d\Omega. \end{aligned} \tag{3.202}$$

The sum of the first term in the integrals in the right-hand part of the equality (3.202) and of the last integral in the equality (3.107) gives zero (this statement follows from the condition (3.196)).

We now prove that

$$\nabla_r \delta u_s = G_s \cdot (\delta u)_{,r}. \tag{3.203}$$

We use the following transformation:

$$\begin{aligned} G_s \cdot (\delta u)_{,r} &= G_s \cdot (G^p \delta u_p)_{,r} = G_s \cdot \left(\frac{\partial G^p}{\partial a^r} \delta u + G^p \frac{\partial \delta u_p}{\partial a^r} \right) \\ &= G_s \cdot \left(-\Gamma^p_{ir} G^i \delta u_p + G^p \frac{\partial \delta u_p}{\partial a^r} \right) = \frac{\partial \delta u_s}{\partial a^r} \Gamma^p_{sr} \delta u_p. \end{aligned} \tag{3.204}$$

It is the last expression of the results obtained that is the covariant derivative of the covariant components δu_s standing on the left-hand side of the equality (3.203) (see the formula (3.159)).

Let us transform the integrand in the last integral (3.202) using the formula (3.203). Taking into account the symmetry $\tau^{ij} = \tau^{ji}$, we obtain

$$\tau^{rs} \nabla_r \delta u_s = \tau^{rs} G_s \cdot \delta u_{,r} = 0{,}5 \tau^{rs} (G_s \cdot \delta u_{,r} + G_r \cdot \delta u_{,s}). \tag{3.205}$$

Since

$$\varepsilon_{ij}^G = \frac{1}{2}(G_i \cdot G_j - \delta_{ij}), \quad G_i = \frac{\partial}{\partial a^i}(a+u) = k_i + u_{,i}, \tag{3.206}$$

therefore

$$\delta\varepsilon_{ij}^G = \frac{1}{2}(G_i \cdot \delta G_j + G_j \cdot \delta G_i) = \frac{1}{2}(G_i \cdot \delta u_{,j} + G_j \cdot \delta u_{,i}). \tag{3.207}$$

Then

$$\tau^{rs}\nabla_r \delta u_s = \tau^{rs}\delta\varepsilon_{rs}^G. \tag{3.208}$$

Assuming now $W = W(\varepsilon_{rs}^G)$ and putting the representations (3.202) and (3.208) into (3.201), we obtain the equality

$$\int_\Omega \frac{\rho}{\rho_0}\frac{\partial W}{\partial \varepsilon_{ij}^G}\delta\varepsilon_{ij}^G \, d\Omega = \int_\Omega \tau^{rs}\delta\varepsilon_{rs}^G \, d\Omega, \tag{3.209}$$

from which, using the arbitrariness of the choice of the domain Ω, we obtain the required formula

$$\tau^{ij} = \frac{\rho}{2\rho_0}\left(\frac{\partial W}{\partial \varepsilon_{ij}^G} + \frac{\partial W}{\partial \varepsilon_{ji}^G}\right). \tag{3.210}$$

(We use the widely used notation, where, by assumption, ε_{ji}^G and ε_{ij}^G are independent if $i \neq j$ and they are connected via the symmetry relation.) Notice that $\rho/\rho_0 = G^{-1/2}$. This relation follows from the formula (3.171). Then the mass conservation law can be written as follows:

$$\rho dV = \rho_0 dV_0. \tag{3.211}$$

Using other representations of the stress tensors $\hat{t}$, $\hat{t}_0$ (see the formulae (3.183), (3.186), and (3.187)), we can find other governing equations as well, e.g., let us consider the representation:

$$\hat{t}_0 = t_0^{ij} k_i \otimes k_j. \tag{3.212}$$

Substituting $x = x(a)$ in the equality (3.201), we obtain

$$\int_{\Omega_0} \delta W \, d\Omega_0 = \int_{\Sigma_0} t_0^{(\nu)} \cdot \delta u \, d\Sigma_0 + \int_{\Omega_0} \rho_0 \left(F - \frac{dv}{dt}\right) \cdot \delta u \, d\Omega_0. \tag{3.213}$$

Using the formulae (3.185) and (3.212), we have

$$\int_{\Sigma_0} t_0^{(\nu)} \cdot \delta u \, d\Sigma_0 = \int_{\Sigma_0} \nu_{0i} t_0^{ij} \delta u_j \, d\Sigma_0 = \int_{\Omega_0} \left(\frac{\partial t_0^{ij}}{\partial a^i}\delta u_j + t_0^{ij}\frac{\partial \delta u_j}{\partial a^i}\right) d\Omega_0. \tag{3.214}$$

Using the result (3.214) in the equation (3.213), the governing equation (3.194) and assuming that

$$W = W(u^j_{,i}), \tag{3.215}$$

we obtain

$$t_0^{ij} = \frac{\partial W}{\partial u^j_{,i}}, \quad u = u^j k_j, \quad u^j_{,i} = \frac{\partial u^j}{\partial a^i}. \tag{3.216}$$

Similar reasonings permit us to find the relation

$$\pi^{ij} = \frac{1}{\sqrt{G}} \frac{\partial W}{\partial u^j_{,i}}, \tag{3.217}$$

$$s_0^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon^G_{ij}} + \frac{\partial W}{\partial \varepsilon^G_{ji}}\right). \tag{3.218}$$

The expression for t_E^{ij} follows from the formula (3.192) and contains the unknown governing equation $x = x(a)$.

If the material is incompressible, the governing equations are modified. In this case we take into account that the admissible displacements satisfy the condition of incompressibility, having the form (see the formula (3.171))

$$\sqrt{G} = 1. \tag{3.219}$$

Using the relation

$$G = \frac{1}{6} E^{ijk} E^{rst} G_{ir} G_{js} G_{kt} \tag{3.220}$$

and the formula for the elements G^{ij} of the inverse matrix $\|G_{ij}\|^{-1}$ defined by the equation

$$G^{ij} G_{jk} = \delta^i_k, \tag{3.221}$$

we find that the field of the admissible displacements for the incompressible material satisfies the condition

$$G^{ij} \delta\varepsilon^G_{ij} = 0. \tag{3.222}$$

Let p be the Lagrange multiplier corresponding to the constraint (3.222) (and the expression (3.222) is equal to the work of the kinematical restriction (3.219)). Using the condition of incompressibility, adding to the left-hand side of the equation (3.209) the term

$$\int_\Omega p G^{ij} \delta\varepsilon^G_{ij} \, d\Omega, \tag{3.223}$$

and repeating the earlier reasoning, we obtain the governing equation for the incompressible material:

$$\tau^{ij} = p G^{ij} + \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon^G_{ij}} + \frac{\partial W}{\partial \varepsilon^G_{ji}}\right). \tag{3.224}$$

Moreover,

$$s_0^{ij} = pG^{ij} + \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ij}^G} + \frac{\partial W}{\partial \varepsilon_{ji}^G}\right). \tag{3.225}$$

Expressions for the components t_E^{ij}, t_0^{ij} in case of incompressibility can be obtained on the basis of the connection between these components and the components τ^{ij}, s_0^{ij}. For example, from the formula (3.190) we have

$$t_0^{ip} = \left[pG^{ij} + \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ij}^G} + \frac{\partial W}{\partial \varepsilon_{ji}^G}\right)\right] x^p_{,j} = pG^{ij} x^p_{,j} + \frac{\partial W}{\partial u^j_{,i}}. \tag{3.226}$$

3.6.5 Principle of virtual work

Assume that the form of the function W is known (some examples will be given later). Consider the following system of equations (below only static problems are investigated):

$$\frac{\partial}{\partial a^i}[s_0^{ir}(\delta_r^j + u^j_{,r})] + \rho_0 F^j = 0, \tag{3.227}$$

$$s_0^{ir}(\delta_r^j + u^j_{,r})\nu_{0i}|_{\Sigma_{0\sigma}} = P^j(x(a))\sqrt{GG^{rs}}\nu_{0r}\nu_{0s} \equiv kP^j, \tag{3.228}$$

$$u|_{\Sigma_{0u}} = 0, \tag{3.229}$$

which describes the strain–stress state of the domain Ω under the action of the prescribed surface tractions with density P and volumetric forces with density ρF. Choose the variables $a^i \in \Omega_0$ as the independent variables.

To simplify the investigation, assume that the set Σ_{0u} is nonempty. Notice that the difficulties arising in the case of the absence of the fixed part Σ_{0u} of the boundary are the same as in the BVPs of the linear theory of elasticity, see Section 2.3.6. In contrast to the linear elasticity problems and to problems of the deformation theory of plasticity, we cannot use, in general, the Hilbert spaces. It is necessary to consider spaces of solutions V of type $W^{1,p}(\Omega)$ (see the definition (1.54)) and sometimes even more complicated spaces the choice of which depends on the structure of the potential W and on the form of the external actions. The question of the choice of the solution space for geometrically nonlinear problems is discussed in [Bal77]. We do not investigate this question here, and in future we will assume that $V \subset W^{1,p}(\Omega)$. This will be the case, e.g., for the polynomial approximation of elastic potential and external actions.

We now deduce the variational equation for the domain Ω_0. Let $W = W(\varepsilon_{ij}^G)$. Then

$$dW \equiv \delta W = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon_{ri}^G} + \frac{\partial W}{\partial \varepsilon_{ir}^G}\right)(\delta_r^j + u^j_{,r})\delta u^j_{,i} = s_0^{ir}(\delta_r^j + u^j_{,r})\delta u^j_{,i}. \tag{3.230}$$

From the equation (3.213) (with the domain Ω_0 coinciding with the whole computation domain) and, taking into account the assumption that the inertial forces are equal to zero, we have the equality

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta u_{,i}^j \, d\Omega_0 = \int_{\Sigma_{0\sigma}} kP \cdot \delta u \, d\Sigma_0 + \int_{\Omega_0} \rho_0 F \cdot \delta u \, d\Omega_0$$
$$\forall \delta u = v - u,\ u \in V,\ v \in V, \quad (3.231)$$

where u is the solution of the problems (3.227)–(3.229) (which can be obtained by multiplying the equation (3.227) by δu^j and using the Gauss–Ostrogradski formula). V denotes the solution space which is some subspace of the functions from $W^{1,p}(\Omega)$, satisfying the boundary conditions (3.229), v is the kinematic admissible field of displacements which is infinitesimally small and differs from the field u. The solution of the variational equation (3.231) is called the generalized solution (distribution) of the problem (3.227)–(3.229). In order to formulate the question on the solvability of the equation (3.231) define operator A, which corresponds to the equation (3.231), by the formula

$$\langle A(u), \delta u \rangle = \int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta u_{,i}^j \, d\Omega_0 - \int_{\Sigma_{0\sigma}} kP \cdot \delta u \, d\Sigma_0 - \int_{\Omega_0} \rho_0 F \cdot \delta u \, d\Omega_0. \quad (3.232)$$

Then the problem of the existence and, sometimes, the uniqueness of the solution (3.231) leads to the analysis of the solvability of an operator equation of the form

$$\langle A(u), \delta u \rangle = 0 \quad \forall \delta u = v - u,\ v \in V,\ u \in V. \quad (3.233)$$

Consider three typical practical cases.

Potential operator

Let the operator A for the considered equation have a potential, i.e., according to Section 3.4 the conditions

$$\langle A'(u, \varphi), \Psi \rangle = \langle A'(u, \Psi), \varphi \rangle \quad \forall \varphi \Psi \quad (3.234)$$

hold. In this case the equation (3.233) is equivalent to finding the stationary point of the functional

$$J(v) = \int_0^1 \langle A(tv), v \rangle \, dt = \int_{\Omega_0} W \, d\Omega_0$$
$$- \int_0^1 \left[\int_{\Sigma_{0\sigma}} k(tv) P(tv) \cdot v \, d\Sigma_0 + \int_{\Omega_0} \rho_0 F(tv) \cdot v \, d\Omega_0 \right] dt. \quad (3.235)$$

Since $V \subset W^{1,p}$ is a reflexive Banach space, it is possible to apply the theory developed in Section 3.3 to the problem of existence and uniqueness. To do this, the convexity of the functional $J(v)$ must be checked.

The properties of the second and third terms in the functional (3.235) are defined by both the properties of the prescribed forces and the geometry of the domain Ω_0. Therefore, the analysis of these terms in any concrete problems is a specific task. The properties of the term

$$J_W(v) = \int_{\Omega_0} W \, d\Omega_0 \tag{3.236}$$

depend only on the form of the potential W. There exists an important particular case when P and F are independent of v, since in this case the second and third integrals in (3.235) are linear functionals. Hence, the problem of checking the convexity is reduced to the analysis of the functional (3.236). In such cases it is possible to find, for given approximations to W, more detailed results.

We give some examples of the approximations to W in order to demonstrate problems in the analysis of the properties $J(v)$, in particular, the convexity of the functional (3.236). Notice that, in the simplest case of isotropic material, the potential W depends only on the invariants of the tensor ε_{ij}^G. For the basic invariants the following expressions are usually used:

$$\begin{aligned} I_1 &= \delta_{ij} G_{ij} = 3 + 2\varepsilon_{kk}^G, \\ I_2 &= \delta_{ij} G_{ij} I_3 = 3 + 4\varepsilon_{kk}^G + 2(\varepsilon_{pp}^G \varepsilon_{qq}^G - \varepsilon_{pq}^G \varepsilon_{pq}^G), \\ I_3 &= \det \|G_{ij}\| = G = \det \|\delta_{ij} + 2\varepsilon_{ij}^G\|. \end{aligned} \tag{3.237}$$

The inclusion $V \subset W^{1,p}(\Omega_0)$ is provided by the polynomial approximation of the potential W with respect to the invariants (3.237). Positing, in the general case,

$$W = W(I_1, I_2, I_3), \quad W(0,0,0) = 0,$$

we have (see, e.g., [GA60]):

$$W \approx \sum_{r=0}^{N_r} \sum_{s=0}^{N_s} \sum_{t=0}^{N_t} C_{rst}(I_1 - 3)^r (I_2 - 3)^s (I_3 - 1)^t, \quad C_{000} = 0. \tag{3.238}$$

In the geometrically nonlinear theory of elasticity we use the following particular forms of the expression (3.238):

1. The Neo–Hookean potential (or the Treloar potential)

$$W = W_{NG} = \frac{\mu}{2}(I_1 - 3), \tag{3.239}$$

   i.e., $N_r = 1$, $N_s = 0$, $N_t = 0$, $C_{100} = \mu/2$
2. The Mooney potential

$$W = W_{MU} = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{3.240}$$

3. The Murnagan potential

$$W = W_{MR} = \frac{1}{8}(\lambda+2\mu)(I_1-3)^2 - \frac{\mu}{2}(I_2-2I_1+3), \quad \lambda = \text{const}, \ \mu = \text{const} \tag{3.241}$$

4. The Adamov–Kuznetsov potential (for weakly compressible materials)

$$W_{AK} = \frac{C_1}{2} I_1 \left(1 - \frac{1}{3}D + \frac{2}{9}D^2\right) + \frac{B}{8}D^2, \quad D = I_3 - 1,$$

being the repeated approximation of the Peng–Landel approximation [Ogd84]:

$$W = W_{PL} = \frac{B}{2}\left(\sqrt{I_3} - 1\right)^2 + \frac{\mu}{2} I_1 I_3^{-1/3}. \tag{3.242}$$

For the general form of the approximation (3.238) we have $V \subset W^{1,p}(\Omega_0)$, $p = N_r + 2N_s + 3N_t$,

$$\|v\|_{W^{1,p}(\Omega_0)} = \left(\sum_{|k|\le 1} \int_{\Omega_0} |\partial^k v|^p \, d\Omega_0\right)^{1/p}. \tag{3.243}$$

Due to the criterion (3.49), the proof of convexity is reduced to checking the inequality:

$$\langle A'(u,v), v\rangle > 0 \quad \forall v \in V, \ u \in V, \ v \ne 0. \tag{3.244}$$

If P and F are independent of v, the following relation holds:

$$\langle A'(u,v), v\rangle = \int_{\Omega_0} \left[\frac{\partial^2 W(u)}{\partial\varepsilon^G_{ir}\partial\varepsilon^G_{pq}} \frac{1}{2}(v^p_{,q} + v,p^q + u^k_{,p}v^k_{,q} + u^k_{,q}v^k_{,p})(\delta_{jr} + u_{j,r}) + \frac{\partial W}{\partial\varepsilon^G_{ir}} v^j_{,r}\right] v^j_{,i} \, d\Omega_0. \tag{3.245}$$

This yields

$$\frac{\partial W}{\partial\varepsilon^G_{ij}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial\varepsilon^G_{ij}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial\varepsilon^G_{ij}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial\varepsilon^G_{ij}}, \tag{3.246}$$

$$\frac{\partial I_1}{\partial\varepsilon^G_{ij}} = 2\delta_{ij}, \quad \frac{\partial I_2}{\partial\varepsilon^G_{ij}} = 4\delta_{ij} + 4[(\varepsilon^G_{kk})\delta_{ij} - \varepsilon_{ij}], \tag{3.247}$$

$$\frac{\partial I_3}{\partial\varepsilon^G_{ij}} = E^{ipq}E^{jrs}(\delta_{pr} + 2\varepsilon^G_{pr})(\delta_{qs} + 2\varepsilon^G_{qs}) \tag{3.248}$$

It follows from the relations (3.245)–(3.248) that the integrand in (3.245) is a quadratic form

$$B^{qs}_{pr} v^p_{,q} v^r_{,s}, \tag{3.249}$$

with coefficients B_{pr}^{qs} which depend on ∇u. The positive definiteness of this form ensures the strict convexity of the functional J_W, and from here we obtain the proof of the existence and uniqueness of the solution for the minimization problem of the functional (3.235) on V.

There are some other ways to ensure convexity, but these involve difficult analysis. There are some hypotheses useful for applications.

Truesdell–Ericksen condition An increase in one of the principal stresses, with the others fixed, corresponds to an increase in the principal elongation.

Backer–Ericksen condition If we compare two different principal stresses, then the larger principal elongation corresponds to the larger stress.

Adamov–Kuznetsov condition To increase the isolated volume of a material with two constant principal stresses, we must decrease the third principal stress and, to decrease the volume, we must increase the third principal stress.

One can also use other conditions of the same type. However, their connection to the condition of convexity remains an open question.

Note that, for incompressible materials, the unknown solution u, and its variation δu have to satisfy the incompressibility conditions. We can remove this condition by use of a Lagrange multiplier, which results in a new term

$$\int_{\Omega_0} pG^{ij}\delta\varepsilon_{ij}^G \, d\Omega_0$$

in the equation (3.231) and a new term

$$\int_{\Omega_0} pG \, d\Omega_0 \tag{3.250}$$

in the functional (3.235). After such a transformation, there appears an additional unknown scalar function p, and the proper theory of solvability, which is based on the results of Section 3.3, is no longer valid. This deficiency appears because of setting of the incompressibility condition, since we are looking for a solution in the subspace of the space V, which is defined by the nonlinear differential equation $G = 1$. There is a theory of solvability for this problem, which is based on theorems relating to saddle points of functionals. We discuss it in Chapter 5.

Analysis in the absence of potential

Suppose that the potential condition does not hold. In this case we consider the equation (3.232) directly, using the theorems of solvability of the operator equations (see, e.g., [Lio69]). We now formulate one such theorem ([Lio69, Theorem 8.2]).

Theorem 3.12. *Assume that in the inequality*

$$\langle A(u), v-u\rangle \geq \langle f, v-u\rangle \quad \forall v \in K \subset V,\ u \in K,\ f \in V^* \tag{3.251}$$

the operator A has the following properties:

(i) $A : V \to V^*$.
(ii) It holds

$$\frac{\langle A(v), v-v_0\rangle}{\|v\|_V} \longrightarrow +\infty \tag{3.252}$$

by $\|v\|_V \to +\infty$ for some $v_0 \in K$, $v \in K$.
(iii) For each $u, v, w \in V$ the function of the real variable λ,

$$\lambda \longrightarrow \langle A(u+\lambda v), w\rangle,$$

is continuous.
(iv) For each $u, v \in V$ we have

$$\langle A(u) - A(v), u-v\rangle \geq 0 \tag{3.253}$$

and the set K is convex and closed in V.

Then there exists at least one solution u of the inequality (3.251) for any $f \in V^*$.

This theorem covers our needs with some "reserve," as we are now examining not an inequality but an equality. Moreover, $f = 0$, $K = V$. The requirement (i) is a condition for the form of the left-hand part. The requirement (iii) can be checked by using the continuity of the functionals in the form of multiple (not more than three) integrals. The most essential conditions are the conditions (ii) and (iv), both of which follow from the inequality

$$\langle A'(u,v), v\rangle \geq \|v\|_V \chi(\|v\|_V) \quad \forall u, v \in V, \tag{3.254}$$

where $\chi(t) > 0$, $t \in \mathbb{R}$ is a real variable,

$$\lim_{t \to +\infty} \chi(t) = +\infty. \tag{3.255}$$

The inequality (3.254) is more general than the condition of convexity in the form (3.244). The condition (3.252) is called the *coercivity of the operator A* and the property (3.253) is called *monotonicity*. If the inequality (3.253) with $u \neq v$ is strict, then the operator A is called *strictly monotonic*. The property of monotonicity is a generalization of convexity when there is no potential. As with strict convexity, strict monotonicity also implies the uniqueness of the solution [Lio69].

Using the formulae (3.245)–(3.248), we can see that in our case the problem of solvability leads us to the analysis of the algebraic properties of the bilinear form (3.249).

Dependence of the solution on the load

The results of the previous two cases correspond to cases where the solution u depends only on the final values of the external actions and is independent of the loading history. However, in the nonlinear theory of elasticity one can give examples where quite different stress–strain states of the body correspond to the same external action. A classic example is the problem of binding of a bar or turning inside out of a tube, where the solution is defined by the given history of the loading.

The practical construction of the solution of such problems is fulfilled by means of a step-by-step procedure. The theoretical investigation is executed by passage to the limit with the infinitesimally small values in the changes of the external actions and in solution. This procedure uses the ordinary differential equations in Banach space for which there is a proper theory of solvability.

We now give the equations for the increments, limit differential equations with respect to the parameter defining the variation in loads and formulate some theorems which allow the possibility of proving the "global" solvability of the problem, i.e., theorems on the existence of the solution to the given history of the load on body.

Choose, as a starting point, the equation (3.233) with the operator A defined by (3.232). Assume that the external actions depend on a parameter t, $0 \le t \le T$. Suppose that if $t = 0$, then the external actions and the solution are zero. Define a partition of the segment $[0, T]$ into subsegments of the length $\Delta t_k = t_k - t_{k-1}$, $k = 1, \ldots, N$. t_k $(k = 0, \ldots, N)$ are the knots of the partition, $t_0 = 0$. Denote by $u^{(k)} \equiv u(t_k)$ the solution for the given value of the parameter $t = t_k$ and by $\Delta u^{(k)} = u^{(k)} - u^{(k-1)}$ the change in the solution when we go from $t = t_{k-1}$ to $t = t_k$. The linearization of the problem with respect to $\Delta u^{(k+1)}$ gives

$$\langle A(u^{(k)} + \Delta u^{(k+1)}), \delta u^{(k+1)} \rangle = 0, \tag{3.256}$$

and we have

$$A(u^{(k)} + \Delta u^{(k+1)}) \approx A(u^{(k)}) + A'(u^{(k)}, \Delta u^{(k+1)}). \tag{3.257}$$

Since $A(u^{(k)}) = 0$, the approximate equations have the form

$$\langle A'(u^{(k)}, \Delta u^{(k+1)}), \delta u^{(k+1)} \rangle = 0 \quad \forall \delta u^{(k+1)} = v - u^{(k+1)},\ v \in V,\ u^{(k+1)} \in V. \tag{3.258}$$

Note that the load parameter t is not included explicitly, either in the potential W or in stress and displacement. Forces P and F explicitly depend on the parameter t. Therefore, as the loading parameter t we can choose any component or modulus of the vectors P or F. Taking into account all of the above arguments, we can establish the following equation:

$$\begin{aligned}
&\langle A'(u^{(k)}, \Delta u^{(k+1)}), \delta u^{(k+1)}\rangle \\
&= \int_{\Omega_0} \left[\frac{\partial^2 W(u^{(k)})}{\partial \varepsilon^G_{ir} \varepsilon^G_{pq}} \frac{1}{2} \big(\Delta u^{(k+1)}_{,q} + \Delta u^{(k+1)q}_{,p} + u^{(k)m}_{,p} \Delta u^{(k+1)m}_{,q} \right. \\
&\left. + u^{(k)m}_{,q} \Delta u^{(k+1)m}_{,p} \big) \left(\delta_{jr} + u^{(k)}_{j,r} \right) + \frac{\partial W(u^{(k)})}{\partial \varepsilon^G_{ir}} \Delta u^{(k+1)}_{j,r} \right] \delta u^{(k+1)j}_{,i} \, d\Omega_0 \\
&- \int_{\Sigma_{0\sigma}} \left[\frac{\partial P^k_0}{\partial u^{(k)p}} \Delta u^{(k+1)p} + \frac{\partial P^{(k)}_0}{\partial u^{(k)p}_{,q}} \Delta u^{(k+1)p}_{,q} + \frac{\partial P^{(k)}_0}{dt} \Delta t^{(k+1)} \right] \cdot \delta u^{(k+1)} \, d\Sigma_0 \\
&- \int_{\Omega_0} \left[\frac{\partial F^{(k)}_0}{\partial u^{(k)p}} \Delta u^{(k+1)p} + \frac{\partial F^{(k)}_0}{\partial u^{(k)p}_{,q}} \Delta u^{(k+1)p}_{,p} + \frac{\partial F^k_0}{dt} \Delta t^{(k+1)} \right] \cdot \delta u^{(k+1)} \, d\Omega_0,
\end{aligned} \tag{3.259}$$

where

$$P^i_0 \equiv (kP)(u^{(i)}, \nabla u^{(i)}, t^i), \quad F^i_0 \equiv (\rho_0 F)(u^{(i)}, \nabla u^{(i)}, t^i).$$

The hypothesis that the external actions depend only on u, ∇u and t cover all practical cases, e.g., the case with the dependence of the external actions on the rotation angles in the neighborhood of a given point of the body.

If the state $u^{(k)}$ corresponding to the load parameter value $t_k \equiv t^k$ is known, then for the calculation of $\Delta u^{(k+1)}$ in the next step we have the linear problem (3.258). To investigate the existence and uniqueness of the solution for this step, we can use the methods which were applied earlier to the linear problem. In particular, we can check the existence of a potential and, where there is a potential, arrive at a problem of functional minimization.

Dividing the equation (3.259) by Δt^{k+1} and putting Δt^{k+1} equal to zero, we arrive at a differential equation in Banach space V:

$$\left\langle \hat{\Phi}^{(k)} \cdot \frac{\partial u^{(k+1)}}{\partial t}, \delta u^{(k+1)} \right\rangle + \left\langle \hat{\Psi}^{(k)} \frac{\partial \nabla u^{(k+1)}}{\partial t}, \delta u^{(k+1)} \right\rangle = \left\langle \theta^{(k)}, \delta u^{(k+1)} \right\rangle$$
$$\forall \delta u^{(k+1)} = v - u^{(k+1)}, \; v \in V, \; u^{(k+1)} \in V, \tag{3.260}$$

where

$$\begin{aligned}
\left\langle \hat{\Phi}^{(k)} \cdot \frac{\partial u^{(k+1)}}{\partial t}, \delta u^{(k+1)} \right\rangle &= - \int_{\Omega_0} \left(\rho_0 \frac{\partial F^{(k)}_0}{\partial u^{(k)p}} \frac{\partial u^{(k+1)p}}{\partial t} \right) \cdot \delta u^{(k+1)} \, d\Omega_0 \\
&- \int_{\Sigma_{0\sigma}} \left(\frac{\partial P^{(k)}_0}{\partial u^{(k)p}} \frac{\partial u^{(k+1)}}{\partial t} \right) \cdot \delta u^{(k+1)} \, d\Sigma_0.
\end{aligned} \tag{3.261}$$

The meaning of the other notations is clear from the comparison of the formulae (3.259), (3.260), (3.261).

The equation (3.261) is used for numerical solution of the problem. The equations (3.260) are well known in mathematics [Kre71]. Many of their properties will be investigated now. These equations differ from the finite system of ordinary differential equations, e.g., by the absence of Peano-type theorems.

To solve these equations we must use digitization with respect to the space variables (i.e., by use of the finite element method (FEM), see, e.g., the monograph [Kre71]), which leads to a Cauchy problem for a finite number of ordinary differential equations with the initial condition $u(0) = 0$. To solve the Cauchy problem, we use either standard methods (like Runge–Kutta or Adams-type) or some special methods and a computer package.

The equation (3.258) corresponds to the simplest case, namely, to the Euler method. Transition to the finite system of differential equations allows us to investigate some other important questions, such as stability analysis, and possible bifurcation of the solution.

4

Unilateral Constraints and Nondifferentiable Functionals

4.1 Introduction: systems with finite degrees of freedom

4.1.1 Example

Consider the problem of the equilibrium of a point mass m in the vertical gravity field with acceleration g on the plane curve. We assume that the point mass moves freely, without friction, on the part AB of the curve (Figure 4.1).

At the ends A and B there are vertical walls limiting the movement of the mass. For an arbitrary internal point C we have the following equation of the virtual displacements principle:

$$R_x \delta x + R_y \delta y = 0. \tag{4.1}$$

The reaction Q of the curve AB does not work on the admissible displacement. The components δx and δy of the virtual displacement δr are limited by the constraint

$$y + \delta y = \varphi(x + \delta x). \tag{4.2}$$

From (4.2) it follows that

$$\delta y = \varphi'(x)\delta x. \tag{4.3}$$

Suppose that the curve ABC is as it is shown in Figure 4.2. The quantity δx in (4.3) is an arbitrary variation. Then at the equilibrium point C we have the equation

$$\varphi'(x) = 0. \tag{4.4}$$

Consider now the situation shown in Figure 4.1. Kinematically admissible displacement of the mass M satisfies the constraint

$$\delta x \equiv x - x_A \geq 0, \tag{4.5}$$

where x_A is the abscissa of the end A of the considered curve.

By supposition, there is no friction on the wall and on the curve. Then the work of the reactions Q and R_x for a kinematically admissible displacement

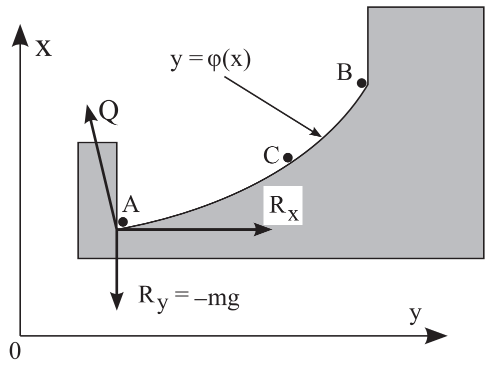

Fig. 4.1. Point mass on a curve with the vertical wall

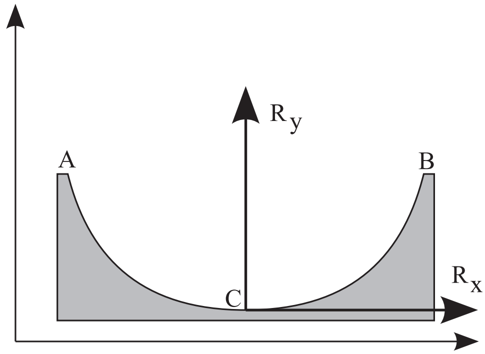

Fig. 4.2. Point mass on a smooth curve

is zero. Taking into account the constraint (4.5) and the equality $R_y = -mg$, we obtain

$$-mg\varphi'(x_A)\delta x \leq 0, \tag{4.6}$$

which is the first example of the *variational inequality*. This inequality defines the equilibrium state of the system. If $\varphi'(x_A) \geq 0$, then the point A is an equilibrium point.

We will meet such kind of inequality for systems with many degrees of freedom and a continuum. We emphasize that the physical meaning of the corresponding variational inequalities will be similar to that just formulated.

4.1.2 Comment of the development of the variational inequalities method

Constraints on the virtual displacements of the system with n degrees of freedom are generally inequalities of the form

$$\delta M_j \equiv \sum_{i=1}^{n} \alpha_{ij} \delta q_j \geq 0, \quad j = 1, \ldots, r, \tag{4.7}$$

where $\alpha_{ij} = \alpha_{ij}(q_1, \ldots, q_n), q_i$ are the generalized coordinates. These constraints are called unilateral (or one-sided) constraints and the corresponding systems are systems with unilateral constraints.

A historical survey of the theory can be found in [Ost61]. Lagrange, the founder of analytical mechanics, did not investigate systems with unilateral constraints. This gap was filled by C. O. Fourier, who gave the appropriate principle for steady-state problems. An analogous formulation was given by Gauss for the equilibrium problem of liquid in a capillary (without reference to the work by Fourier).

The complete theory of systems with unilateral constraints was given by M. V. Ostrogradski [Ost61]. He was the first to investigate some of the problems with unilateral constraints for continuum systems (liquids) where he used the Lagrange multipliers method. With this method, the equilibrium equations (in the variational form) include the additional terms:

$$\sum_{j=1}^{r} \delta M_j \mu_j. \tag{4.8}$$

One of the important results consists of the demonstration of the theorem about the signs of the Lagrange multipliers μ_j. The sign of a multiplier is opposite to that of the quantity δM_j (Ostrogradski investigated the constraints (4.7) and the constraints $\delta M_j \leq 0$).

The essential result of M. V. Ostrogradski consists in the construction of the motion equations for systems with unilateral constraints. These systems are now known in analytical mechanics as systems with nonholding (or liberating) bonds. Ostrogradski proposed an algorithm for the solution that takes into account the bonds until the Lagrange multiplier changes sign. It must follow the sign of a bond in time: if the sign changes we must discard this bond in the corresponding D'Alambert equations deduced with the Lagrange multiplier method.

The only incorrect proposition made by M. V. Ostrogradski was in the formulation of the initial set of active bonds. He proposed to analyze the unilateral constraint

$$f_i(x_1, y_1, z_1, x_2, \ldots, z_n; t) \geq 0, \quad i = 1, \ldots, r, \tag{4.9}$$

by decomposition of the function f_i into a series with respect to the infinitesimal increment of time dt in the neighborhood of the motion beginning at t_0:

$$f_{i0} + f'_{i0} dt + f''_{i0} \frac{(dt)^2}{2} + \ldots \geq 0 \tag{4.10}$$

with

$$
\begin{aligned}
f_{i0} &\equiv f_i(x_{10}, \ldots, z_{n0}; t), \\
f'_{i0} &\equiv \left(\frac{\partial f_i}{\partial t}\right)_0 + \left(\frac{\partial f_i}{\partial x_1}\right)_0 x'_{10} + \ldots + \left(\frac{\partial f_i}{\partial z_n}\right)_0 z'_{n0}, \\
f''_{i0} &\equiv \left(\frac{\partial f_i}{\partial x_1}\right)_0 x''_{10} + \ldots + \left(\frac{\partial f_i}{\partial z_n}\right)_0 z''_{n0} + A_i,
\end{aligned}
\tag{4.11}
$$

where (x_i, y_i, z_i) are the coordinates of the point number i of the system and (x_{i0}, y_{i0}, z_{i0}) are the initial coordinates of this point. The quantities A_i are calculated from the initial values of the coordinates and velocities. Following Ostrogradski, we must take into account at the initial time only those of the r bonds for which the following inequalities hold simultaneously:

$$f_{i0} = 0, \quad f'_{i0} = 0, \tag{4.12}$$

$$f''_{i0} = 0. \tag{4.13}$$

The first constraint in (4.12) disappears for nonholonomic bonds.

The problem is that the quantities (4.13) depend on the initial acceleration values and they are unknown. Ostrogradski proposed to liberate the system from all the bonds, to find the acceleration values for the free system and find the signs of the quantities f''_{i0} with these acceleration values. The conclusion was that we must to take into account bonds with nonpositive values of the quantities f''_{i0}. This proposition was incorrect and was revised by A. Mayer. Mayer's solution consists in the determination of the initial acceleration values from the principle of the smallest constraint of Gauss. In compliance with this principle the quantity

$$Z = \sum_{j=1}^{r} m_i \left[\left(x''_i - \frac{F_{ix}}{m_i}\right)^2 + \left(y''_i - \frac{F_{iy}}{m_i}\right)^2 + \left(z''_i - \frac{F_{iz}}{m_i}\right)^2\right] \tag{4.14}$$

must be the minimal on the real motion of the system. Comparison is made for all the motions allowed by the bonds imposed on the system. From r unilateral constraints (4.9) we must analyze all their combinations. At the initial time t_0 we must take into account the combination for which the values of the quantity Z are minimal in comparison with any other set of unilateral bonds. The existence theorem was proved by Mayer for the value $r = 1, 2$. The general case with the arbitrary value of r was given by Zermelo, using the methods of multidimensional geometry.

Later, mathematical problems similar to the static and dynamical problems for systems with unilateral constraints were found in many new branches of science, technology, and economics. These developments can be divided into two approaches. The first is mathematical programming (generalization of the "static" problems) and the second is the so-called control problems ("dynamical" generalization). The novelty of those problems consists of finding the optimal project (analogously to the problem of the equilibrium state) or to find the optimal control for the moving system (similarly to the dynamic problem for systems with unilateral constraints). The system under investigation is described by means of different sets of variables. There are the control variables, state variables, and the observations, dependent on the state and control variables. The observations are included in the cost function (or functional) which we must minimize or maximize. Both the state variables and the control variables can be submitted to the constraints, as they are inequalities and analogous to the nonholding bonds. Problems without the time variable are those of mathematical programming theory. With the time variable these problems become control problems. This classification is rough but useful for an initial orientation.

Historically, the first problem was an example from mathematical programming theory – so-called linear programming. This theory was developed in the pioneer investigations of the L. V. Kantorovich [Kan39] on economic problems. A general approach to the linear programming problem, the simplex method was developed by J. Danzig in 1947.

The natural generalization of linear programming was nonlinear programming, in which both cost function and constraints are in general nonlinear. The necessary optimality conditions were first formulated by H. W. Kuhn and A. W. Tucker [KT51]. Later developments of the theory and methods of nonlinear programming were made by K. J. Arrow, L. Hurwitz, H. Uzawa, and others (see [AHU58]).

Mathematical theory, as the generalization of these results on continuum systems, is the basis for the most of the problems investigated in our book: parts of this theory were given above in Sections 3.2–3.4. Additional information will be given in the sections below in the solutions to partial problems.

Modern advances concern many dynamic problems. Based on this investigations are the fundamental results of L. S. Pontryagin, R. Bellmann, and their followers [PBGM64].

4.2 Variational methods in contact problems for deformed bodies without friction

In this section the theory developed in Sections 3.2–3.4 and the methods mentioned above will be applied to continuum systems with unilateral constraints. We will investigate deformed bodies in contact with unilateral constraints. In this book we use the approach "from simple problems to more and more complicated ones." Hence we begin with the investigation of the contact between one deformed body and a rigid stamp and move on to the more complicate problems involving the contact between arbitrary systems of deformed bodies.

4.2.1 Contact problems for a beam and membrane

Consider a beam of length l with rigid supports on the ends. The elasticity modulus of the beam E and the second moment of area is I. Choose the Cartesian coordinate system in the plane of the bending of the beam (Figure 4.3). In general, $E = E(x)$, $I = I(x)$.

Consider the bending of the beam in the plane Oxy by a rigid stamp without friction. The stamp surface is given by the equation

$$\Psi(x, y) = 0. \tag{4.15}$$

Suppose that, in the initial (nondeformed) state, the stamp touches the beam at one (or a few) points and that for this state we have the equation (4.15). We impose the restriction that at a point (x, y) $\Psi(x, y) > 0$ outside the stamp and $\Psi(x, y) < 0$ inside the stamp. The stamp is assumed to be convex. (This hypothesis is not necessary, see below.)

Move the stamp downwards over the distance U_0. The equation of the surface of the stamp after this movement takes the form

$$\Psi(x, y + U_0) = 0. \tag{4.16}$$

The closed system of equations describing the stress and strains distributions across the beam include

- The equilibrium equation:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 w}{dx^2}\right] = q(x) \tag{4.17}$$

- The boundary conditions:

$$w(0) = w(l) = 0, \qquad \left.\frac{dw}{dx}\right|_{x=0} = \left.\frac{dw}{dx}\right|_{x=l} = 0 \tag{4.18}$$

- The condition of the impenetrability of the surface of the beam inside the stamp:

$$\Psi(x, w + U_0) \geq 0 \tag{4.19}$$

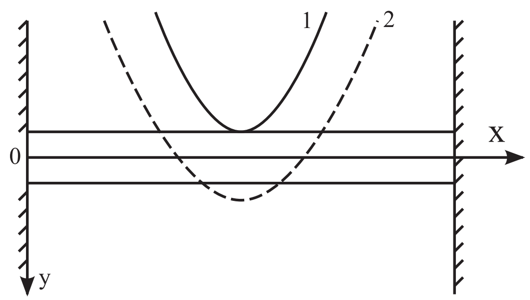

Fig. 4.3. Contact of a beam with a rigid stamp

- The condition of the reaction nonnegativity of the stamp:

$$q(x) \geq 0 \tag{4.20}$$

- The equation:

$$\Psi(x, w(x) + U_0)q(x) = 0, \quad 0 < x < l, \tag{4.21}$$

  which describes the requirement that a strict equality in one (or two) of the conditions (4.19) and (4.20) holds at any point x.

The unknowns are the displacement $w(x)$ of the middle line of the beam and the force $q(x)$ interacting between the stamp and beam. Unknown is also the domain in which the condition $\Psi(x, w(x) + U_0) = 0$ holds – this is the unknown contact zone.

The inequality (4.19) is an unilateral constraint – a liberating (or nonholding) bond. More precisely, the set of unilateral constraints at all the points $x \in (0, l)$ is a continuum constraint. For the transition to the variational setting of the problem we use the functional space $V = H^2(0, l)$, see the definition (1.57).

With the reasoning used in deducing the equation (2.63) we find that the equation (4.17) with the boundary condition (4.18) is equivalent to the variational equation (with appropriate restrictions on the smoothness of the solution)

$$\int_0^l EI(x)w''\delta w''dx = \int_0^l q(x)\delta w\, dx \quad \forall \delta w \equiv v - w,\ v \in V,\ w \in V. \tag{4.22}$$

Suppose that the function $\Psi(x)$ is continuously differentiable, and investigate the sign of the product $q(x)\delta w(x)$. At points where there is no contact between the beam and stamp this product is zero. If at the point x we have contact then

$$\Psi(x, w + U_0) = 0, \quad \Psi(x, v + U_0) \geq 0. \tag{4.23}$$

Since $v = w + \delta w$ where the function δw is infinitesimally small, from (4.23) we have the following restriction on the variations δw:

$$\left(\left.\frac{\partial \Psi(x, y)}{\partial y}\right|_{y=w+U_0} \right) \delta w \geq 0 \quad \forall x \in (0, l). \tag{4.24}$$

The above hypothesis on the function $\Psi(x, y)$ gives:

$$\left.\frac{\partial \Psi(x, y)}{\partial y}\right|_{y=w+U_0} > 0. \tag{4.25}$$

Then from the inequalities (4.24) and (4.20) it follows that

$$q(x)\delta w(x) \geq 0 \quad \forall x \in (0, l). \tag{4.26}$$

Hence, the solution $w(x)$ of the problem (4.17)–(4.21) satisfies the variational inequality

$$\int_0^l EI(x)w''\delta w''dx \geq 0 \quad \forall \delta w = v - w,\ v \in K,\ w \in K \subset V, \tag{4.27}$$

where K is the set of kinematically admissible fields of the deflection

$$K = \{v \mid v \in V;\ \Psi(x, v(x) + U_0) \geq 0\ \forall x \in (0, l)\}. \tag{4.28}$$

We have not proved the inverse transition from the variational inequality (4.27) to the local problem (4.17)–(3.23) yet. To make this transition, suppose that the solution $w(x)$ has the fourth derivative and integrate the left-hand side of the inequality (4.27) twice by parts. Using the conditions (4.18) (which hold according to the hypothesis), we find that

$$\int_0^l (EI(x)w'')''\delta w\, dx \geq 0. \tag{4.29}$$

We investigate two cases – without contact and with contact. In the first case we have

$$\begin{aligned} \Psi(x, w(x) + U_0) + \delta w(x)\frac{\partial \Psi(\ldots)}{\partial y} &\geq 0, \\ \Psi(x, w(x) + U_0) &> 0. \end{aligned} \tag{4.30}$$

Then the variation $\delta w(x)$ can have any sign. Using the same method as in the demonstration of an analogous statement in Chapter 2, we find that

$$(EI(x)w'')'' = 0. \tag{4.31}$$

Let the beam touch the stamp at some point x. For this point we will have

$$\frac{\partial \Psi(\ldots)}{\partial y}\delta w \geq 0, \tag{4.32}$$

whence with the inequality (4.25) it follows that

$$\delta w(x) \geq 0. \tag{4.33}$$

Assuming the inverse statement, we demonstrate that the following equation holds:

$$(EI(x)w'')'' \equiv q(x) \geq 0. \tag{4.34}$$

The condition (4.34) simultaneously gives the equation (4.17) as a way of calculating the reaction q from the known solution $w(x)$ of the inequality (4.27) and the nonnegativity of the reaction (4.20).

Furthermore, for the transformations the method developed in Sections 3.2–3.4 is employed. Introducing, first, the operator $A(w)$ by the formula

$$\langle A(w), v\rangle = \int_0^l EI(x)w''(x)v''(x)\,dx \quad \forall v \in V, \tag{4.35}$$

we demonstrate that this operator has the potential

$$J(v) = \frac{1}{2}\int_0^l EI(x)(v'')^2 dx. \tag{4.36}$$

Using the results of Section 2.4, we find that the functional (4.36) is strongly convex. Restoring the hypothesis on the convexity of the function $\Psi(x, y)$ and using Theorem 3.10 (the inequality (3.73)), we see that the inequality (4.27) is equivalent to the problem of the minimization of the functional (4.36) on the convex set K. The existence and uniqueness of the solution to this problem follows from Theorems 3.9 and 3.10.

Note that the hypothesis on the convexity of the function $\Psi(x, y)$ can be replaced by the appropriate smoothness requirements of this function since the impenetrability condition (4.19) defining the admissible set K for the small deflections can be linearized in the variable w:

$$\Psi(x, U_0) + \left.\frac{\partial\Psi(x, y)}{\partial y}\right|_{y=U_0} \quad w(x) \geq 0. \tag{4.37}$$

Recall that the hypothesis on the smallness of the deflection w and its derivatives was used in the deduction of the equation (4.17). Replacing the impenetrability condition in the definition of the set K by the linearized one (4.37) means that the set K will be convex for all smooth functions $\Psi(x, y)$. This result permits the transition from the variational inequality (4.27) to the minimization problem for the functional (4.36) on the convex set K as well as applications of the existence and uniqueness theorems to the solution of the given problem.

Note that the Kirchhoff–Lowe hypothesis together with the Hertz method of the gap calculation leads to the fact that the contact stresses reduce to a set of concentrated forces, i.e., to a sum of the Dirac delta functions.

Indeed, H. Hertz uses the first two members of the Taylor series of the function which describes the gap δ between two contacting bodies [Gla80]

$$\delta(x, y) = \delta_0 + Ax^2 + By^2, \tag{4.38}$$

where $A = \text{const}$, $B = \text{const}$, $\delta_0 = \text{const}$, and x and y are the coordinates of a point in the plane tangent to the initial contact point. By supposition, initial contact point is alone, contacting bodies are convex, and origin O of the Cartesian coordinate system Oxy is at the initial contact point.

In the beam contact problem the boundary of a rigid stamp is given by (4.16). We suppose that the initial contact point is $x_c = l/2$ and that the difference $(x - l/2)$ is small. Following the Hertz method, we obtain the gap equation

$$y(x) = \delta(x) = \varphi(l/2) + 1/2\varphi''_{xx}|_{x=l/2}(x - l/2)^2, \tag{4.39}$$

where φ''_{xx} is the second derivative with respect to x.

In a contact point the first of the equations (4.23) holds. Then, with the Hertz method we have the following formula for the deflection $w(x)$ in the contact domain:

$$y(x) = w(x) = \varphi(l/2) + 1/2\varphi''_{xx}|_{x=l/2}(x - l/2)^2, \tag{4.40}$$

and from the beam equation (4.17) we obtain the equality $q = 0$.

Therefore, the contact stresses are reduced to

$$q(x) = \sum_i P_i \delta(x - x_i), \tag{4.41}$$

where P_i is a concentrated force at the point $x = x_i$ and $\delta(x)$ is the Dirac function. Firstly, this statement was formulated in [Gal48].

The coordinates x_i are determined by the condition (4.20): we calculate the concentrated moment at the point x_i and, taking into account the fact that a moment is produced by force couple with the opposite forces (one of these forces is a stretching force), we equate this couple zero. This method was proposed first in [EMT69].

Note that the variational method permits the solution of a contact problem without the Hertz decomposition and determines the continuous distribution of contact stresses.

We now consider the problem of membrane bending where the deflection is limited by a rigid fixed obstacle. The surface of the obstacle is described by the equation

$$z = \psi(x, y). \tag{4.42}$$

This problem is now more complicated because the domain Λ of the independent spatial variables is 2D. Let the boundary Γ of the membrane be fixed, and let the membrane be loaded by the external pressure p in the direction orthogonal to the middle plane. Let $Oxyz$ be the Cartesian coordinate system such that the domain Λ is in the plane Oxy and the deflection is oriented along the axis Oz.

The local setting of the problem for the determination of the deflection $w(x, y)$ and the reaction $q(x, y)$ consists of the following:

- The equilibrium equation:

$$-\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \equiv -\Delta w = p - q \tag{4.43}$$

- The boundary condition:

$$w|_\Gamma = 0 \tag{4.44}$$

- The impenetrability condition:

$$w(x, y) \leq \psi(x, y) \tag{4.45}$$

- The hypothesis on the nonnegativity of the reaction:

$$q(x,y) \geq 0 \tag{4.46}$$

- The equation:

$$q(x,y)[w(x,y) - \psi(x,y)] = 0, \tag{4.47}$$

which is analogous to the equation (4.21)

By means of the procedure used in the previous problem we can demonstrate that the problems, (4.43) and (4.47) is equivalent to the variational inequality

$$\int_{\Lambda} \left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} - p\delta w \right) dx\, dy \geq 0 \quad \forall \delta w = v - w,\ v \in K,\ w \in K, \tag{4.48}$$

where

$$K = \{v \mid v = v(x,y),\ (x,y) \in \Lambda;\ v \in H^1(\Lambda);\ v|_{\Gamma} = 0; \\ v(x,y) \leq \psi(x,y)\ \forall (x,y) \in \Lambda\}. \tag{4.49}$$

Using the potentiality of the operator A defined by the relation

$$\langle A(w), v\rangle = \int_{\Lambda} (\nabla w \cdot \nabla v - pv)\, dx\, dy \quad \forall v \in K, \tag{4.50}$$

the convexity of the potential $\Psi(v)$ corresponding to this operator

$$J(v) = \frac{1}{2}\int_{\Lambda} |\nabla v|^2 dx\, dy - \int_{\Lambda} pv\, dx\, dy, \tag{4.51}$$

and Theorem 3.10 (Section 3.3), we deduce that the variational inequality (4.48) is equivalent to the minimization problem of the functional (4.51) on the convex set (4.49) and that the solution to this problem exists and is unique.

4.2.2 Contact between an elastic body and a rigid stamp

Let the deformed solid occupy the domain Ω with the boundary $\Sigma = \Sigma_u \cup \Sigma_\sigma \cup \Sigma_C$. For the internal points of the domain Ω the equilibrium equations (2.228) hold. On the parts Σ_u and Σ_σ of the boundary we have the boundary conditions (2.229) and (2.230). The points on the part Σ_C of the boundary touch the surface of the stamp. Let the surface stamp equation be

$$\Psi(x) = 0, \quad x = (x_1, x_2, x_3). \tag{4.52}$$

Suppose that, as before, $\Psi(x) < 0$ inside and $\Psi(x) > 0$ outside the stamp. Then the boundary conditions on the part Σ_C are the following:

- The impenetrability condition:

$$\Psi(x + u(x)) \geq 0 \tag{4.53}$$

- The hypothesis on the nonpositivity of normal pressure:

$$\sigma_N \equiv (\hat{\sigma} \cdot \nu) \cdot \nu \leq 0 \tag{4.54}$$

- The absence of friction:

$$\sigma_T \equiv \hat{\sigma} \cdot \nu - \sigma_N \nu = 0 \tag{4.55}$$

- The equations:

$$\Psi(x + u(x))\sigma_N(x) = 0 \quad \forall x \in \Sigma_C, \tag{4.56}$$

  which are analogous to the equations (4.21) and (4.47).

Recall the notation: $u(x)$ is the displacement of the deformed body, supposed to be infinitesimally small, $\hat{\sigma} = \sigma_{ij} k_i \otimes k_j$ is the stress tensor, and ν is the unit normal vector orthogonal to the surface Σ.

Suppose that the function Ψ is twice differentiable and the second derivative of this function is bounded. Since the displacements $u(x)$ are small, we can linearize the impenetrability condition (4.53) with respect to the function u:

$$\Psi(x + u(x)) \approx \Psi(x) + u(x) \cdot \nabla\Psi(x) \geq 0. \tag{4.57}$$

In the theoretical considerations it will be useful to use the following simplification of the impenetrability condition:

$$u \cdot \nu \equiv u_N \leq \delta_N \approx \frac{\Psi(x)}{|\nabla\Psi(x)|}. \tag{4.58}$$

Simplification consists of the use of the approximate equality

$$\nabla\Psi(x) \approx -|\nabla\Psi(x)|\nu. \tag{4.59}$$

Below we demonstrate that the errors introduced by the conditions (4.57) and (4.58) are both of the second order. In applications we can use any of the impenetrability conditions.

For the transition to the variational setting in the space V defined as $V = [H_0^1(\Omega)]^n \equiv H_0^1(\Omega)$, we replace the uniform boundary condition on the whole boundary Σ by the following one:

$$v|_{\Sigma_u} = 0. \tag{4.60}$$

Define the closed convex set K:

$$K = \{v \mid v \in V;\ v_N(x) \le \delta_N(x)\ \forall x \in \Sigma_C\}. \tag{4.61}$$

Suppose (temporarily) that the reaction $\hat{\sigma} \cdot \nu$ of the stamp is known and repeat the demonstration used in the transition to the variational inequality (2.231). We obtain the equation

$$a(u, \delta u) = L(\delta u) + \int_{\Sigma_C} (\hat{\sigma} \cdot \nu) \cdot \delta u \, d\Sigma, \tag{4.62}$$

where

$$a(u, \delta u) = \int_{\Omega} \hat{\sigma}(u) \cdot \cdot \hat{\varepsilon}(\delta u) \, d\Omega, \tag{4.63}$$

$$L(\delta u) = \int_{\Omega} \rho F \cdot \delta u \, d\Omega + \int_{\Sigma_\sigma} P \cdot \delta u \, d\Sigma, \tag{4.64}$$

$$\hat{\varepsilon}(v) = \varepsilon_{ij}(v) k_i \otimes k_j,$$

$$\varepsilon_{ij}(v) = (\partial v_i / \partial x_j + \partial v_j / \partial x_i)/2. \tag{4.65}$$

Note that we did not use the special hypothesis on the dependence of the stress tensor $\hat{\sigma}$ on the strain tensor $\hat{\varepsilon}(u)$. This fact permits us to assert that the above results are valid both for linear materials (with the Hooke law governing equation $\hat{\sigma} =^4 \hat{a} \cdot \hat{\varepsilon}(u)$) and for nonlinear ones, e.g., materials with the governing equation (3.125).

For the transition to the variational problem we give the following two theorems.

Theorem 4.1. *The problem (2.228)–(2.230), (4.53)–(4.56) is equivalent to the variational inequality*

$$a(u, \delta u) \ge L(\delta u) \quad \forall \delta u = v - u,\ u \in K,\ v \in K. \tag{4.66}$$

Proof. Write the expression to the integral for the variational equation (4.62) as follows:

$$(\hat{\sigma} \cdot \nu) \cdot \delta u = \sigma_T \cdot \delta u_T + \sigma_N \delta u_N, \tag{4.67}$$

where the index N indicates that the corresponding quantity is the orthogonal projection onto the unit outward normal vector ν. The index T indicates the projection onto the tangent plane (orthogonal to the vector ν).

Assume that there is no friction, i.e., $\sigma_T = 0$. Then, if the impenetrability condition holds as a strong inequality, it follows that $\sigma_N = 0$. If $u_N = \delta_N$ then the condition $v \in K$ gives the inequality $v_N \le \delta_N$. So $\delta u_N \equiv v_N - u_N \le 0$. Taking into account the inequality (4.54), we finally obtain

$$(\hat{\sigma} \cdot \nu) \cdot \delta u \ge 0 \quad \forall x \in \Sigma_C. \tag{4.68}$$

From this result we obtain the first of the assertions of the theorem: all the solutions of the problem (2.228)–(2.230), (4.53)–(4.56) are the solutions of the variational inequality (4.66).

Now let the function u be a solution of the variational inequality (4.66). It is easy to demonstrate that the equilibrium equation (2.228) and the conditions (2.230), (4.54), and (4.55) hold. The demonstration is as follows. Consider a set $D(\Omega)$ of functions with a support compact in the domain Ω (any function from the set $D(\Omega)$) is equal to zero with all their derivatives on boundary Σ and in a neighborhood of the boundary. In fact, in the problem in hand it is sufficient that the functions are twice differentiable. First substituting in the inequality (4.66) the sum $u + \varphi$ and the differences $u - \varphi$, $\varphi \in D(\Omega)$, we obtain the equation

$$a(u, \varphi) = L(\varphi) \quad \forall \varphi \in D(\Omega). \tag{4.69}$$

Transforming the left-hand part of this equation with the Green formula (2.219), we obtain the equilibrium equation (2.217) (taking into account the density of the embedding of the set $D(\Omega)$ in all Sobolev spaces).

Now multiply the equilibrium equation by a variation $\delta u \equiv v - u$, $v \in K$, $u \in K$, and integrate the product over the domain Ω. Using once again the Green formula, we obtain

$$a(u, \delta u) = \int_{\Omega} \rho F \cdot \delta u \, d\Omega + \int_{\Sigma_\sigma} (\hat{\sigma} \cdot \nu) \cdot \delta u \, d\Sigma + \int_{\Sigma_C} (\hat{\sigma} \cdot \nu) \cdot \delta u \, d\Sigma. \tag{4.70}$$

Subtracting this equation from the inequality (4.66), we obtain

$$\int_{\Sigma_\sigma} [\hat{\sigma}(u) \cdot \nu - P] \cdot \delta u \, d\Sigma + \int_{\Sigma_C} [\hat{\sigma}(u) \cdot \nu] \cdot \delta u \, d\Sigma \geq 0. \tag{4.71}$$

Let the element v in this inequality satisfy the additional restriction $v = u$ on Σ_C. Then

$$\int_{\Sigma_\sigma} [\hat{\sigma}(u) \cdot \nu - P] \cdot \delta u \, d\Sigma \geq 0 \quad \forall \delta u = v - u. \tag{4.72}$$

Now we must use some theorems on the trace spaces. For complete definitions and demonstrations of the theorems concerning the spaces of the boundary traces of a function, see, e.g., [Neč67, CR80] (see also Theorem 1.96).

So, let φ be the trace of a function $\delta u = v - u$ at the boundary Σ_σ. Consider the inequality (4.72) for a case where the function φ belongs to the set $D(\Sigma_\sigma)$ of functions which are infinitely differentiable with a support compact in Σ_σ. We find that

$$\int_{\Sigma_\sigma} [\hat{\sigma}(u) \cdot \nu - P] \cdot \varphi \, d\Sigma = 0 \quad \forall \varphi \in D(\Sigma_\sigma). \tag{4.73}$$

Using the density of the embedding $D(\Sigma_\sigma)$ into $L^2(\Sigma_\sigma)$, we obtain the equality

$$\hat{\sigma}(u) \cdot \nu - P = 0 \tag{4.74}$$

i.e., on Σ_σ.

Using the inequality (4.71), the equation (4.74) and linearity and continuity of the map $H^1(\Omega) \to L^2(\Sigma)$, we obtain the inequality

$$\int_{\Sigma_C} [\hat{\sigma}(u) \cdot \nu] \cdot \delta u \, d\Sigma \geq 0 \quad \forall v \in K. \tag{4.75}$$

Using the decomposition (4.67) and arbitrary choice of the variation δu_T with $\delta u_N = 0$, we demonstrate that the equation

$$\int_{\Sigma_C} \sigma_T(u) \cdot \delta u_T \, d\Sigma = 0 \quad \forall \delta u_T \in D(\Sigma_C), \;\; \delta u_N = 0, \tag{4.76}$$

holds from which, with the density of the embedding $D(\Sigma_C)$ into $L_2(\Sigma_C)$, we obtain the condition (4.55). (Equality means that there is a zero element in the space $L_2(\Sigma_C)$.)

From the equation (4.76) and above continuity of the map $H^1(\Omega)$ in $L_2(\Sigma)$ pointed out it follows that the inequality (4.71) has the form

$$\int_{\Sigma_C} \sigma_N(u) \delta u_N \, d\Sigma \geq 0 \quad \forall \delta u = v - u, \;\; v \in K, \;\; u \in K. \tag{4.77}$$

Now decompose the boundary Σ_C into two subsets:

1. Σ_C^v consisting of the contact points
2. $\Sigma \setminus \Sigma_C^v$

and apply the method used earlier twice to demonstrate that

$$\sigma_N(x) = 0 \quad \forall x \in \Sigma_C \setminus \Sigma^v. \tag{4.78}$$

Then

$$\int_{\Sigma_C^v} \sigma_N(u) \delta u_N \, d\Sigma \geq 0 \quad \forall \delta u = v - u, \;\; v \in K, \;\; u \in K. \tag{4.79}$$

By the method of contradiction we demonstrate that, from the inequality (4.79), it follows the inequality

$$\sigma_N(u(x)) \leq 0 \quad \forall x \in \Sigma_C^v, \tag{4.80}$$

from which, together with the equation (4.78), we obtain the condition (4.54). The theorem is proved.

Theorem 4.2. *The inequality (4.66) is equivalent to the minimization of the functional*

$$J(v) = \frac{1}{2} a(v, v) - L(v) \tag{4.81}$$

on the subset K. A solution to the minimization problem exists and is unique.

Proof. The proof reduces to the verification of the supposition of Theorem 3.11 and of the potentiality operator theorem. The essential hypotheses, on the strict convexity of the functional (4.81) and potentiality of the operator $A(u)$ defined by the formula

$$\langle A(u), v\rangle = a(u, v) - L(v) \quad \forall v, \tag{4.82}$$

have been demonstrated earlier. The convexity of the set K is evident because this set is defined by linear constraints. The closedness of K in the space $H^1(\Omega)$ follows from the continuity of the map from the space $H^1(\Omega)$ into the trace space, see, e.g., [Neč67]. Hence, the theorem is demonstrated.

4.2.3 Contact of a system of deformed bodies

Consider a set of bounded deformed bodies $\Omega^1, \ldots, \Omega^M$ with boundaries $\Sigma^1 \equiv \partial\Omega_1, \ldots, \Sigma^M \equiv \partial\Omega^M$. Recall that the symbol $\partial\Omega, \partial\Sigma, \ldots$ means the boundary of the domain $\Omega, \Sigma, \ldots$. Suppose that in the initial state (without strains and stresses) the domains are in contact at some points or on parts of their boundaries. The structures of these boundaries are defined by the initial geometry. To simplify the notations below, we will omit (if it is not confusing) the index indicating the given numbers of bodies. In the contact between two bodies we will label them Ω, Ω' with surfaces Σ, Σ', etc.

Suppose that the boundary Σ of a body consist of three parts: $\Sigma = \Sigma_u \cup \Sigma_\sigma \cup \Sigma_C$. On the parts Σ_u and Σ_σ the classic boundary conditions are given: displacements and surface tractions are prescribed. As a variant we can assign the adjacency conditions

$$u|_{\Sigma_u} = u'|_{\Sigma'_u}, \quad \hat{\sigma} \cdot \nu|_{\Sigma_\sigma} + \hat{\sigma}' \cdot \nu'|_{\Sigma_\sigma} = 0. \tag{4.83}$$

Assume that the shape and dimensions of the maximal contact domains for the body Ω are known from geometrical and mechanical analysis. The union of such domains will be denoted by Σ_C. Note that this hypothesis concerns only the limit contact domain. The real contact domains are unknown and must be determined by the solution of the boundary value problem. Suppose also that $\partial\Sigma_C = \Sigma_C \cap \Sigma_\sigma$. This hypothesis, in practice, usually holds and will be used below.

As earlier, denote the density of the prescribed volume forces by ρF and the density of surface tractions by P. Then the general problem is to define the strains and stresses for all the touching bodies Ω under the given forces $P^1, \ldots, P^M$, $\rho F^1, \ldots, \rho F^M$. We must also find the shapes and dimensions of the contact domains and distributed pressure in the contact domains. Below we construct the mathematical model of this problem as a variational inequality and a functional minimization problem under the inequalities constraints. Friction phenomena will be omitted.

The main constraint on the point displacements on S_c consists of the hypothesis that the material points of one of the contacting bodies cannot penetrate inside the other. This is the "impenetrability condition." Let the equations for the contact surfaces Σ_C and Σ'_C of the bodies Ω and Ω' be the following:

$$\Psi(x) = 0, \quad \Psi'(x') = 0. \tag{4.84}$$

Suppose that the functions Ψ, Ψ' satisfy the hypotheses

$$\begin{aligned} \Psi(x) > 0, &\quad \text{if } x \text{ is outside } \Omega, & \Psi(x) < 0, &\quad \text{if } x \text{ is inside } \Omega, \\ \Psi'(x') > 0, &\quad \text{if } x' \text{ is outside } \Omega', & \Psi'(x') < 0, &\quad \text{if } x' \text{ is inside } \Omega'. \end{aligned} \tag{4.85}$$

Suppose now that we can linearize the impenetrability condition with respect to all the geometrical parameters – displacements, their derivatives, the gap between surfaces (boundaries), curvatures of boundaries. Executing elementary but bulky calculations [Kra78], we obtain the result that the impenetrability condition for the bodies Ω, Ω' can be used in several different equivalent forms, e.g.,

$$u_N(x) - u'_N(x') \leq \delta_N, \tag{4.86}$$

where u_N is the orthogonal projection of the displacement of point $x \in \Sigma_C$ of the current body onto the external normal ν to this body, u'_N is the displacement projection of the point $x' \in \Sigma'_C$ onto the normal the body Ω at point x, x' is the intersection point of the segment going from the point x in the direction of the vector ν, with the surface Σ_c, and δ_N is the length of this segment between Σ_C and Σ'_C (Figure 4.4).

In the impenetrability condition the bodies Ω, Ω' can be exchanged one to another. The result is the following:

$$u'_{N'}(x') - u_{N'}(x) \leq \delta_{N'}, \tag{4.87}$$

where $u_{N'} = u \cdot \nu'$, $u'_{N'} = u' \cdot \nu'$, ν' is the normal to the body Ω'. The correspondence between the points x and x' and the gap definition is illustrated by Figure 4.4 (see the left-hand side).

Using the equation (4.84), we can find the relation between the coordinates of the points x and x'. First, we find the root $t = t_0$ of the equation

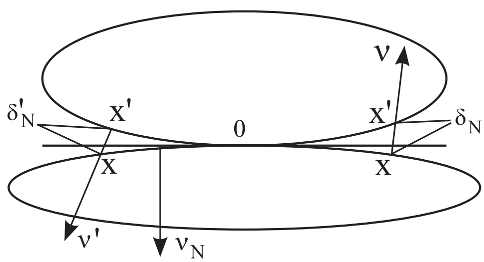

Fig. 4.4. The definition of the gap between two deformed bodies

$$\Psi'(x + t\nabla\Psi(x)) = 0 \tag{4.88}$$

corresponding to the point $x' \in \Sigma'$ at the outward unit normal to Σ. So, in the impenetrability conditions we use

$$x'(x) = x + t_0\nabla\Psi(x). \tag{4.89}$$

If the equation (4.88) allows linearization with respect to the variable t then

$$t_0 = \Psi'(x)[\nabla\Psi(x) \cdot \nabla\Psi'(x)]^{-1}. \tag{4.90}$$

An analogous procedure gives the expression for the variable x and x' in the condition (4.87). The gaps in the above impenetrability conditions are:

$$\delta_N = |x - x'(x)|, \quad \delta_{N'} = |x(x') - x'|. \tag{4.91}$$

Notice that it is valid to use displacement projections of the points at the surfaces Σ_C and Σ'_C onto the normal to an arbitrary plane with the normal different, but "not too much," from the directions defined by the unit vectors ν and ν'. For example, we can use the normal to the common tangential plane at the initial contact point O. This method is used in the Hertz theory (Figure 4.4). The term "not too much" means that the error of the conditions (4.86) and (4.87) due to linearization is of the second order with respect to the displacements, first derivatives of the displacements and gaps δ_N, $\delta_{N'}$. Any impenetrability condition with the second order precision can be used in the linearized theory. For example, we can use the Hertz impenetrability condition

$$u_N^H(x) - {u'}_N^H(x') \leq \delta_N^H, \tag{4.92}$$

where index "H" means that we use the projection of the displacements onto the normal to the plane tangent to the initial contact point (by supposition there is a unique initial contact point and the contact surfaces are smooth).

Transition to the variational inequality and to the minimization problem is justified by the following two theorems.

Theorem 4.3. *The problem on the determination of stresses and strains for a system of contacting bodies is equivalent to the variational inequality*

$$a(u, \delta u) \geq L(\delta u) \quad \forall \delta u = v - u,\ v \in K,\ u \in K \subset \bar{V}, \tag{4.93}$$

where

$$a(u, \delta u) = \sum_{I=1}^{M} a^I(u^I, \delta u^I), \tag{4.94}$$

$$L(\delta u) = \sum_{I=1}^{M} L^I(\delta u^I). \tag{4.95}$$

The functionals $a^I(u^I, v^I)$ *and* $L^I(v^I)$ *are defined by the formulae (4.63), (4.64) for all the bodies* Ω^I $(I = 1, 2, \ldots, M)$ *of the system under consideration.* $\bar{V}$ *is the direct product of the spaces defined as a subspaces of* $H^1(\Omega^I)$*, with the appropriate forced boundary conditions,* $v = \{v^1, v^2, \ldots, v^M\}$*. K is the set*

$$K = \{v \mid v \in \bar{V};\ v_N^I(x^I) - v_N^J(x^J(x^I)) \leq \delta_N^{IJ}\}. \tag{4.96}$$

We use here the impenetrability condition (4.86), and explicit the numbers "I" and "J" of the contacting bodies. Inequalities in the definition (4.96) of set K refer to all the parts Σ_C *of the contacting body boundaries.*

Proof. Write the equation of the virtual displacements for each body of the system – this equation has the form (4.62). Add these equations together. The result is

$$a(u, \delta u) = L(\delta u) + \sum_I \int_{\Sigma_C^I} [\hat{\sigma}(u) \cdot \nu] \cdot \delta u \, d\Sigma. \tag{4.97}$$

Decompose the last sum in this equation into pairs corresponding to two bodies in contact with touching surfaces Σ_C and Σ_C'. Consider one pair in more detail

$$A \equiv \int_{\Sigma_C} [\hat{\sigma}(u) \cdot \nu] \cdot \delta u \, d\Sigma + \int_{\Sigma_C'} [\hat{\sigma}(u') \cdot \nu'] \cdot \delta u' \, d\Sigma'. \tag{4.98}$$

Using the formula (4.67) and the zero-friction hypothesis, we see that

$$[\hat{\sigma}(u) \cdot \nu] \cdot \delta u = \sigma_N(u) \delta u_{N'}, \quad [\hat{\sigma}(u') \cdot \nu'] \cdot \delta u' = \sigma_N' \delta u_N'. \tag{4.99}$$

Recall that in the linearized theory the approximate relation

$$\nu \approx -\nu', \quad \sigma_N(u) \approx -\sigma_N(u'). \tag{4.100}$$

holds. The same hypothesis permits us to omit the difference between Σ_C and Σ_C' in the calculation of the surface integrals in (4.98). Then

$$A = \int_{\Sigma_C} \sigma_N(u) [\delta u_N(x) - \delta u_N(x'(x))] \, d\Sigma. \tag{4.101}$$

If some point $x \in \Sigma_C$ is a noncontact point, then at this point the integrand is equal to zero. If at the point $x \in \Sigma_C$ there is contact then $u_N(x) - u_N(x'(x)) = \delta_N$. Since

$$v_N(x) - v_N(x'(x)) \leq \delta_N,$$

then

$$\delta u_N(x) - \delta u_N(x'(x)) \leq 0. \tag{4.102}$$

As earlier, we use the hypothesis: the normal component σ_N of the surface tractions is compressive. Then

$$\sigma_N(x) \leq 0 \quad \forall x \in \Sigma_C. \tag{4.103}$$

From the inequalities (4.102) and (4.103) it follows that at every point $x \in \Sigma_C$ the integrand in the integral (4.101) is nonnegative. Then we demonstrate that any solution of the boundary value problem consisting of the equilibrium equations for all the bodies Ω^I $(I = 1, \dots, M)$ under boundary conditions

$$u|_{\Sigma_u} = 0, \quad \hat{\sigma} \cdot \nu|_{\Sigma_\sigma} = P, \tag{4.104}$$

$$u_N(x) - u_N(x'(x)) \leq \delta_N(x), \tag{4.105}$$

$$\begin{cases} \sigma_N(x) \leq 0, \quad \sigma_T(x) = 0, \\ \sigma_N(x)[u_N(x) - u_N(x'(x)) - \delta_N(x)] = 0 \quad \forall x \in \Sigma_C \end{cases} \tag{4.106}$$

satisfies the variational inequality (4.93). Hence, Theorem 4.3 is demonstrated.

The inverse transition from the variational inequality (4.93) to the local equilibrium equations with the corresponding boundary conditions is realized as in the previous problem. In correspondence with the well-known terminology the boundary conditions (4.106) and condition on Σ_σ are called *natural*. The kinematic restriction (4.105) and the boundary condition on Σ_u define the admissible displacement set K and are called *forced* conditions.

If we need to find the strains and stresses on some internal surfaces Σ_u, e.g., on the joint surfaces of materials with different mechanical properties, then we must add the condition (4.83) to the set of the kinematically admissible displacements K. This constraint is an equality, it does not violate the convexity of the set K, and does not change the proof of Theorem 4.3.

Theorem 4.4. *The variational inequality (4.93) is equivalent to the minimization problem for the functional*

$$J(v) = \frac{1}{2} a(v, v) - L(v), \quad v \in K. \tag{4.107}$$

The notations are the same as in Theorem 4.3.

Proof. Consider two essentially different cases. In the first one, every body has a clamped nonempty boundary part Σ_u. In the second case, one body or several bodies (or all the bodies) do not have any clamped boundary points.

In the first case, it follows from the Korn inequality that for each linear elastic body Ω^I of the given system there is positive definiteness (see also the inequalities (2.262)–(2.272))

$$a(u^I, u^I) \geq c_I \|u^I\|^2_{V^I}, \quad c_I = \text{const} > 0. \tag{4.108}$$

Taking into account the symmetry of the operator

$$\langle A(u), v \rangle = a(u, v) - L(v), \tag{4.109}$$

Theorem 3.9, and the theorem on potentiality, we obtain a statement on the equivalence of the variational inequality (4.93) to the minimization problem of the functional (4.107) and on the existence and uniqueness of the solution. Only one additional definition must be given – the definition of the norm $v \in V$ by the formula

$$\|v\|_V^2 = \sum_I \|v^I\|_{V^I}^2, \tag{4.110}$$

where $\|v^I\|_{V^I} = \|v^I\|_{H^1(\Omega^I)}$.

In the second case, with the absence of a fixed part of the boundary for one or several contacting bodies the expression (4.110) will be only a seminorm on the space V. There are nonzero displacement fields where the expression (4.110) is zero.

The general laws of mechanics permit us to formulate the following statements:

1. The system of the contacting bodies must be in equilibrium: the resultant external force and moment must be zero:

$$\sum_I \left[\int_{\Omega^I} \rho F^I \, d\Omega + \int_{\Sigma_\sigma^I} P^I \, d\Sigma \right] = 0, \tag{4.111}$$

$$\sum_I \left[\int_{\Omega^I} x \times \rho F^I \, d\Omega + \int_{\Sigma_\sigma^I} x \times P^I \, d\Sigma \right] = 0. \tag{4.112}$$

2. The position of each body in the current state of the system must be stable. This means that a little rigid "shaking" of these bodies from their initial state and from the current state demands expenditure of energy. In other words, the work of the prescribed external forces on the corresponding displacement fields must be strictly negative.

These preliminary arguments were first formulated by Signorini for the equilibrium of an elastic body in a rigid envelope, i.e., with the boundary condition $u_N \le 0$, $u_N = u \cdot \nu$, ν is outwards normal drawn to the boundary. Later, these statements were generalized as theorems. The most general theorems are due to J.-L. Lions and G. Stampacchia [Lio69].

Theorem 4.5 (Lions–Stampacchia theorem). *For inequality (4.93), let the following hypotheses hold:*

(i) The bilinear form $a(u, v)$ is continuous on the space V:

$$|a(u, v)| \le c\|u\|\|v\| \quad \forall u \in V, \ v \in V, \ c = \text{const} > 0 \tag{4.113}$$

(ii) The norm on V is equivalent to the norm $p_0(v) + p_1(v)$, where $p_0(v)$ is a norm on V and the space V with this norm is a pre-Hilbert[1]. $p_1(v)$ is a seminorm on V.

[1] See the definition in [Ped89, p. 81].

(iii) The space

$$Y = \{v \mid v \in V;\ p_1(v) = 0\} \tag{4.114}$$

is finite-dimensional.

(iv) There exists a constant c_1 *such that*

$$\inf_y p_0(v - y) \leq c_1 p_1(v), \quad \forall y \in Y \cap K \tag{4.115}$$

(v) The bilinear form $a(u, v)$ *is positive semidefinite, i.e.,*

$$a(v, v) \geq c_2 p_1^2(v) \quad \forall v \in V,\ c_2 = \text{const} > 0 \tag{4.116}$$

(vi) The set K *is closed and convex in* V*, and the zero-point of the space* V *belongs to* K*.*

(vii) The linear form $L(v)$ *is continuous on* V*, with*

$$L(y) < 0 \quad \forall y \in Y \cap K \tag{4.117}$$

Then the variational inequality (4.93) has at least one solution.

Proof. Let

$$p_0^2(v) = \sum_I \int_{\Omega^I} v^I \cdot v^I \, d\Omega,$$

$$p_1^2(v) = \sum_I \int_{\Omega^I} \varepsilon_{ij}(v^I)\varepsilon_{ij}(v^I) \, d\Omega, \tag{4.118}$$

$$\|v\|_V^2 = p_0^2(v) + p_1^2(v). \tag{4.119}$$

It follows from the Korn inequality that the norm (4.119) is equivalent to the norm (4.110). The quantity $p_0(v)$ is a norm on the space L^2, the space H^1 with this norm is a pre-Hilbert space, and the same statement is valid for the space V. The quadratic form $p_1^2(v)$ is a seminorm on the space V, because $\varepsilon_{ij}(y) = 0$ for a rigid displacement field y of the body Ω.

The space Y defined by the formula (4.114) is a finite-dimensional space since it is a direct product of the finite-dimensional spaces

$$Y^I = \{y^I \mid y^I = a + b \times x,\ a = \text{const},\ b = \text{const},\ x \in \Omega^I\}. \tag{4.120}$$

The inequality (4.115) is demonstrated by the method used in [DL72]. The positive semidefiniteness of the form $a(u, v)$ follows from the Korn inequality for each body Ω^I. So, all the statements of the Lions–Stampacchia theorem are demonstrated. Note that we demonstrated existence and uniqueness without transition to the minimization problem.

To transform the problem into a minimization problem, we must use the space V defined earlier in Section 2.4. Each element of such a space is defined as a set of displacement fields, and the difference between two displacement fields from the same set is a rigid body displacement. It is this space that permits us to demonstrate the uniqueness of the solution. With this reservation Theorem 4.4 is proved.

Note that in the demonstration of the theorem we established the additional constraints (4.111), (4.112), and (4.117) necessary for the formulation of the problem. Consider in detail the condition (4.117). It easy to see that this condition, known as the "strong Signorini hypothesis," has the form

$$L(y) < 0 \quad \forall y \in Y \cap K \setminus G, \tag{4.121}$$

where G is the displacement field corresponding to the movement of all the contacting bodies $\Omega^1, \ldots, \Omega^M$ seen as a whole rigid body. Elimination of the field G in the condition (4.121) follows immediately from the equilibrium equation as a whole (4.111) and (4.112). The restriction (4.121) means that the work of the prescribed (external) forces on the movement of each body Ω, supposed to be a rigid displacement field in its initial state, is strictly negative. This is the stability requirement mentioned earlier.

Instability of the deformed state equilibrium can be revealed as a deformation effect. Let u be the solution of the problem. Define the set Z by the formula

$$Z = \{y \mid y \in K;\ u + y \in K,\ \forall y \in G\}. \tag{4.122}$$

It follows from the variational equation (4.93) and the conditions (4.111) and (4.112) that

$$L(y) \leq 0 \quad \forall y \in Z \setminus G. \tag{4.123}$$

This condition, as the consequence of the inequality (4.93) and the introduced hypothesis, means that after deformation the system of contacting bodies is in a stable or neutral state. In general, the sets $Y \cap K$ and Z are different, because the condition $u + y \in K$ does not imply that $y \in K$.

In conclusion we make some additional remarks:

1. Theorems 4.3 and 4.4 remain valid for the Hencky–Ilyushin theory of plasticity (see Section 3.5). As well these demonstration of the convexity used the condition (3.134) and the inequality (3.135). The functional mentioned in Theorem 4.4 has the form (3.132).
2. If one of the contacting bodies, e.g., $\Omega \equiv \Omega^J$ is a rigid body then we must add the following restriction to the kinematic displacement field:

$$v(x) = a + b \times x, \quad x \in \Omega,\ a = \text{const},\ b = \text{const}. \tag{4.124}$$

   In practice, the restriction (4.124) can be eliminated with the Lagrange multiplier. The constants a and b are prescribed if we have a moving rigid stamp with the prescribed translation and rotation or one determined from the equation following from the equilibrium requirement of the stamp. Notice that if $a = b = 0$ and $M = 2$ the problem addressed in Section 4.2.2 arises.
3. To find a unique solution for the displacements in the cases where the boundary has no fixed part, we can use the method described above for the noncontact problems with the forces prescribed on the whole of the boundary, i.e., to use the quotient spaces $\widetilde{V}$ (see Section 2.3.4).

4.2.4 Geometrically nonlinear theory of elasticity

As in Section 4.2.2, consider in detail the problem of the contact between a deformed body and a rigid stamp. From the statement of the problem, the displacements as well the strains from the initial state Ω_0 to the deformed state Ω are not assumed to be small (see the essential hypothesis in Section 3.6). We will use the methods and models developed in Section 3.6.

The formulation of the problem contains:

- The equilibrium equation (see (3.195))

$$\frac{\partial}{\partial a^i}[s_0^{ir}(\delta_r^j + u^j_{,r})] + \rho_0 F^j = 0 \tag{4.125}$$

- The governing equation in the form (3.218)

$$s_0^{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \varepsilon^G_{ij}} + \frac{\partial W}{\partial \varepsilon^G_{ji}}\right) \tag{4.126}$$

- The boundary condition with the prescribed displacements

$$u|_{\Sigma_{0u}} = 0 \tag{4.127}$$

- The boundary condition with the prescribed surface tractions (3.200)

$$s_0^{ir}(\delta_r^j + u^j_{,r})\nu_{0i}|_{\Sigma_{0\sigma}} = P^j(x(a))\sqrt{GG^{rs}}\nu_{0r}\nu_{0s} \equiv P^j k \tag{4.128}$$

- The boundary conditions in the contact domain Σ_{0C} (by supposition $\Sigma_{0u} \cup \Sigma_{u\sigma} \cup \Sigma_{0C} = \Sigma_0 \equiv \partial\Omega_0$ is the whole boundary of the body Ω_0):

$$\Psi(x(a)) > 0 \implies t^{(\nu)} = 0, \tag{4.129}$$

$$\Psi(x(a)) = 0 \implies t_N \leq 0, \quad t_T = 0, \tag{4.130}$$

$$\Psi(x(a))t_N(x(a)) = 0 \quad \forall a \in \Sigma_{0C}, \tag{4.131}$$

  where $t^{(\nu)}$ is the density of the surface forces on the boundary Σ of the deformed body Ω related to the density of the conditional surface forces (in the initial state) by the relation (3.184).

  We use the decomposition

$$t^{(\nu)} = t_N \nu + t_T \tag{4.132}$$

analogous to (4.55), where ν is the external unit vector orthogonal to the boundary Σ of body Ω in its deformed state. As in Section 4.2.2, $\Psi(x) < 0$ at inside the stamp, $\Psi(x) > 0$ outside, with the same assumption on differentiability.

The transition to the variational formulation is justified by the following theorems.

Theorem 4.6. *The problem (4.125)–(4.131) is equivalent to the variational inequality*

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta u_{j,i}\, d\Omega_0 - \int_{\Omega_0} \rho_0 F_0 \cdot \delta u\, d\Omega_0 - \int_{\Sigma_{0s}} kP \cdot \delta u\, d\Sigma_0$$
$$\equiv \langle A(u), \delta u\rangle \geq 0 \quad \forall \delta u = v - u,\ u \in K,\ v \in K,\ \delta u \in K_u, \quad (4.133)$$

where[2]

$$K = \{v \mid v(a) = 0,\ a \in \Sigma_{0u};\ \Psi(a + u(a)) \geq 0\ \forall a \in \Sigma_{0C}\}, \quad (4.134)$$

$$K_u = \{\delta u \mid \Psi(a + u(a)) + \delta u \cdot \nabla\Psi(a + u(a)) \geq 0\ \forall a \in \Sigma_{0C};\ \delta u = 0 \text{ on } \Sigma_{0u}\}. \quad (4.135)$$

Proof. Suppose temporarily that the tractions on Σ_{0C} are known with the density $t_0^{(\nu)}$ and write the equation of virtual work:

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta u_{,i}^j\, d\Omega_0 = \int_{\Omega_0} \rho_0 F_0 \cdot \delta u\, d\Omega_0 + \int_{\Sigma_{0\sigma}} kP \cdot \delta u\, d\Sigma_0 +$$
$$+ \int_{\Sigma_{0C}} t_0^{(\nu)} \cdot \delta u\, d\Sigma_0 \quad \forall \delta u \in K_u,\ u \in K,\ v \in K. \quad (4.136)$$

Using the relation between $t_0^{(\nu)}$ and $t^{(\nu)}$ and the decomposition (4.132), write the integrand in the integral on the surface Σ_{0C} in the form

$$t_0^{(\nu)} \cdot \delta u = k(t_N \delta u_N + t_T \cdot \delta u_T) = k t_N \delta u_N. \quad (4.137)$$

The last step in (4.137) is due to the absence of friction. If at a point $a \in \Sigma_{0C}$ the impenetrability condition is a strict inequality then the right-hand part in (4.137) is equal to zero. If $\Psi(a + u(a)) = 0$ then

$$\delta u \cdot \nabla\Psi(a + u(a)) \geq 0. \quad (4.138)$$

Since in this case we have

$$\nabla\Psi(a + u(a)) = -\nu|\nabla\Psi(a + u(a))|,$$

then $\delta u_N \leq 0$ and from the condition (4.130) it follows that the right-hand part of the equality (4.137) is nonnegative. So

$$\int_{\Sigma_{0C}} t_0^{(\nu)} \cdot \delta u\, d\Sigma_0 \geq 0.$$

Then every solution of the problem (4.125)–(4.131) satisfies the variational inequality (4.133). The inverse transition from the variational inequality to the local formulation is made as before. The theorem is proved.

[2] The space V of the solutions u and kinematically admissible displacement v is defined by the potential form W, see Section 3.6.5. The essential difference from the geometrically linear theory consists in the fact that there exists a set of admissible variations K_u depending on the unknown solution u. Such variations are defined by the prescribed forces $\rho_0 F$, P.

The existence and uniqueness theorem of the variational inequality (4.133) was formulated at the end of Section 3.6.

If we try to make the transition to the minimization problem, we meet problems similar to those arising in the usual (noncontact) problem. In particular, the potentiality not only of the operator A introduced by the formula (4.133), but also of the part of the operator A defined by the external actions as well as the strict potential convexity on the whole must be ensured. This problem can be solved very seldom, and then, usually in practice, the variational inequality (4.133) is solved without transition to the minimization problem.

The most difficult and the most interesting problem in applications is the geometrically nonlinear contact problem for two deformed bodies. Such problems arise in the modeling of rubber or in polymer mechanical behavior, in biology, in some medical applications, etc.

4.2.5 Comments

The equilibrium of a continuous system with unilateral constraints was investigated first by A. Signorini [Sig33]. A detailed exposition of these results was given by A. Signorini in the article [Sig59]. The problem is now known as the *Signorini problem.* It consists of determining the stresses and strains of a linear elastic body Ω placed in a rigid smooth shell before loading. In the variational approach, the solution u and the kinematically admissible displacements v are subjected to additional constraint, the inequality

$$v \cdot \nu|_{\partial\Omega} \leq 0, \tag{4.139}$$

where $\partial\Omega$ is the boundary of the deformed body Ω and the vector ν is the outward drawn normal unit vector orthogonal to the boundary $\partial\Omega$.

For half a century there was no effective solution for the Signorini problem, and this problem was investigated only as an interesting theoretical problem from the point of view of partial differential equation theory. The existence and uniqueness theorems were demonstrated (G. Duvaut, G. Fichera, G. Levi, and others). A survey of these results is given in Fichera's monograph [Fic72]. The most important mathematical results were achieved by the Italian mathematician G. Stampacchia. A survey of Stampacchia's results [Lio80] consists of more than 80 publications. Important results were obtained by J.-L. Lions, his disciples and by many Italian and French mathematicians. Note also the monographs [Pan85, KO88].

With the advent of the computer and its use in research, considerable progress has been made in numerical methods for Signorini-type problems. These methods will be described in Chapter 7. At present we mention only the monograph [GLT81] where the fundamental theoretical results for numerical methods are collected.

4.3 Variational method in contact problem with friction

4.3.1 Generalities

The surfaces of deformed solids in contact always contain defects and non-homogeneities even after very careful treatment. These defects and non-homogeneities are the origin of the tangential component of the surface tractions. This fact implies that the hypothesis (4.55) is not valid. Thus we must introduce governing laws for the evolution of tangential stresses at the contact boundary. These laws are called *friction laws.* Historically the first, and widely used, is the Amonthon–Coulomb law, often called the Coulomb friction law. This law relates the tangential components of force and the relative velocity of the bodies in contact.

Later, the essential reasoning and calculations will be performed for the contact of a single deformed body with a fixed rigid stamp. A generalization will be given in Chapter 7. In this case the tangential component of the relative sliding velocity is equal to the absolute sliding velocity. Denote this velocity by $\dot{u}_T \equiv \partial u/\partial t$ where, as usual, the variable t is time or some other parameter defining evolution of the deformed body state. Write the *Coulomb governing equation* as the following set of equations and inequalities:

$$|\sigma_T| \leq f|\sigma_N| \implies \quad \dot{u}_T = 0, \tag{4.140}$$

$$|\sigma_T| = f|\sigma_N| \implies \quad \exists \kappa \geq 0 : \dot{u}_T = -\kappa \sigma_T, \tag{4.141}$$

where f is the friction coefficient. Note that in the slip domain $|\dot{u}_T| > 0$ and

$$\frac{\sigma_T}{|\sigma_T|} = -\frac{\dot{u}_T}{|\dot{u}_T|}. \tag{4.142}$$

This formula will be used in the numerical applications.

The normal component of the contact force must be compressive, i.e., $\sigma_N \leq 0$. The constraint on the tangential component $|\sigma_T|$ can be written as follows:

$$|\sigma_T| \leq -f\sigma_N. \tag{4.143}$$

This inequality must not be violated.

A particular feature of the governing law (4.140) and (4.141) consists of its "threshold": the sliding of one body on the other one begins not for all the values of the tangential component σ_T but only after reaching some threshold value equal to $-f\sigma_N$. The second property of the law (4.140) and (4.141) is the presence of the displacement velocities: the system is nonconservative, i.e., there is dissipation, and the current state depends on the history of the contact.

These aspects of the law (4.140) and (4.141) hinder analytical solutions and make more hard the theoretical investigations of the existence, uniqueness and regularity of a solution. This difficulty was the reason for simplifying the

Coulomb friction law, for example, to eliminate the threshold (4.140). It led to the "liquid" friction law

$$|\sigma_T| = -f\sigma_N. \tag{4.144}$$

The direction of the vector σ_T is defined by the second relation (4.141).

For some problems of the linear theory of elasticity, the direction of the vector σ_T is known *a priori.* In such a case it is possible to eliminate the velocities appearing in the contact problem. To do this, we must introduce some additional displacement hypotheses. For example, we can suppose that the normal components of displacements are equal one to another at the contacting points:

$$u_N^{(1)} = u_N^{(2)}.$$

Sometimes the function $u_N^{(2)}$ is known.

The law governing friction (4.140) and (4.141) has been verified in many experiments. The limits of its validity and numerous generalizations are well known. Some of these generalizations will be described below. In spite of the simplicity and limitations of the law (4.140) and (4.141) it often gives good results for the contact stresses for many industrial problems and is a good test for new theories and methods of solving contact problems with friction. These reasons explain the wide use of the law (4.140) and (4.141).

4.3.2 Quasi-variational inequality for the contact problem with friction

This section is devoted to the transition from a local formulation to the variational one. The basis of this transition is the well-known physical principle – the *principle of maximum dissipation power* for contact problems considered in [CK85, HdSM02]. There exists an analogy between the variational formulation for the law governing plastic flow and the Amonthon–Coulomb friction law. It is possible to obtain the friction law by changing the stress deviator by the tangential effort vector σ_T and by changing the deviator of the strain velocity tensor by the vector $\dot{u}_T$ in the law governing plastic material. The plastic flow surface in the stress space in this analogy is replaced by the "surface" (4.144), etc. The result is that the set of the relations (4.140) and (4.141) is analogous to the law governing ideal plastic material.

In the sequel we will use the inequality

$$-\dot{u}_T \cdot (\tau_T - \sigma_T) \leq 0 \quad \forall \tau_T,\ |\tau_T| \leq -f\sigma_N, \tag{4.145}$$

as the maximum principle for the frictional stresses. The quantity τ_T in the inequality (4.145) is the statically admissible tangential effort, the quantity σ_T is the true tangential effort, and $\dot{u}_T$ is the true relative sliding velocity on the contact surface. More strict definitions and demonstrations of the inequality (4.145) will be given below.

Let us assume the following problem: find the stresses and strains in the domain Ω with the boundary $\Sigma = \Sigma_u \cup \Sigma_\sigma \cup \Sigma_C$ as in Section 4.2.2, replacing the boundary condition (4.55) by the set of the relations (4.140) and (4.141).

Suppose, in addition, that the contact domain and contact interaction stresses σ_N and σ_T are known. Then the set of the equations, conditions and constraints (2.113) (motion equation), (2.11) (the Hooke law), (2.115) (the Cauchy relations), (2.229) and (2.230) (boundary conditions on Σ_u, Σ_σ), and (4.53), (4.54), (4.56), (4.140) and (4.143) (contact condition and friction law), defining the deformed state of the body Ω, are equivalent to the integral identity

$$a(u, \delta\dot{u}) = L(\delta\dot{u}) + \int_{\Sigma_C} (\sigma_N \delta\dot{u}_N + \sigma_T \cdot \dot{u}_T)\, d\Sigma - \int_\Omega \rho\ddot{u} \cdot \delta\dot{u}\, d\Omega, \qquad (4.146)$$

where $\delta\dot{u}$ is the variation in the velocity field under the constraints defined by the Ostrogradski method (see Section 4.1), $a(u, v)$ is the bilinear form defined by (2.222), and $L(u)$ is the linear form (2.232) where Σ is replaced by Σ_σ. Consider the problem of establishing constraints on the admissible velocity fields in dynamics, and after that move on to the quasi-static problems.

Direct transition from the local formulation to the integral identity is performed with the same scheme as for the contact problem without friction, using the Gauss theorem. The inverse transformation can be performed with this method under additional constraints on the variations $\delta\dot{u}$. These variations belong to the space $H^1(\Omega)$ for any value t. Strictly we must investigate the properties of the solution with respect to the independent variable t. We do not go into details on the corresponding definitions referring to the exhaustive investigation [LM73]. We give only the essential definitions which can be found in the monographs by J.-L. Lions [GLT81, Lio68, Lio75].

Let X be a Banach space (in fact, for a linear elastic material we need only a Hilbert space). Notate by $L^p(0, T; X)$ the space of the function $t \to f(t)$ with the range of values in X being strongly measurable on $[0, T]$ with respect to the Lebesgue measure dt on the segment $[0, T]$, with

$$\left(\int_0^T \|f(t)\|_X^p \, dt \right)^{1/p} \equiv \|f\|_{L^p(0,T;X)} < +\infty. \qquad (4.147)$$

We will use only the case $p = 2$. Suppose that $X \equiv V$ where V is a Sobolev space.

The following statement is valid [LM73]: if $f \in L^2(0, T; V)$, then the derivatives $\partial f/\partial t, \partial^2 f/\partial t^2, \ldots$ can be defined as distributions on the domain $(0, T)$ with range of values in V by the formula

$$\left\langle \frac{df}{dt}, \varphi \right\rangle = -\left\langle f, \frac{d\varphi}{dt} \right\rangle \quad \forall \varphi \in \mathcal{D}(0, T; V), \qquad (4.148)$$

where, as usually, $\mathcal{D}$ is the set of infinitely differentiable functions with a compact support in the domain of definition of these functions. Angular brackets mean a linear functional on the space V. We will use the standard notation for the derivative $df/dt \equiv \partial f/\partial t$.

Require that for any solution u of the equation (2.113) (after substitution (2.114) and (2.115)) the conditions

$$u(x,t) \equiv u(t) \in L^2(0,T;V), \quad \frac{\partial u}{\partial t} \in L^2(0,T;V^*) \tag{4.149}$$

hold. (Recall that a star denotes the conjugation operator.) It can be demonstrated that

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;V^*). \tag{4.150}$$

Then as a functional solution space we can choose the space

$$W(0,T) = \left\{ f \mid f \in L^2(0,T;V),\ \frac{\partial f}{\partial t} \in L^2(0,T;V^*) \right\} \tag{4.151}$$

with the norm

$$\|f\|_W = \left(\int_0^T \|f\|_V^2 \, dt + \int_0^T \left\| \frac{\partial f}{\partial t} \right\|_{V^*}^2 dt \right)^{1/2}. \tag{4.152}$$

We turn now to the definition of the set $K \subset W(0,T)$ where we find the solution. To solve this problem, we can use the Ostrogradski method [Ost61]. Suppose temporarily that the velocities and accelerations are not small and write the impenetrability condition for two time moments t_0 and $t_0 + dt$ without linearization of the function $\Psi(x+u)$ with respect to the displacements u:

$$\Psi(x + u(x,t_0)) \geq 0, \quad \Psi(x + u(x, t_0 + dt)) \geq 0. \tag{4.153}$$

Decompose the left-hand part of the second inequality in (4.153) in a series with respect to the variable dt:

$$\begin{aligned} &\Psi(x + u(x,t_0)) + \left(\frac{\partial \Psi}{\partial u_i} \frac{\partial u_i}{\partial t} \right)\Bigg|_{t=t_0} dt \\ &\quad + \left(\frac{\partial^2 \Psi}{\partial u_i \partial u_j} \frac{\partial u_i \partial u_j}{\partial t \partial t} + \frac{\partial \Psi}{\partial u_i} \frac{\partial^2 u_i}{\partial t^2} \right)\Bigg|_{t=t_0} \frac{dt^2}{2} + O(dt^3) \geq 0. \end{aligned} \tag{4.154}$$

For motion of the system beginning at moment t_0, the kinematical parameters will be restrained only in the part $\Sigma_C^{t_0}$ of the boundary Σ_C defined by

$$\begin{aligned} \Sigma_C^{t_0} = \Bigg\{ x \mid \Psi(x + u(x,t_0)) = 0;\ &\left(\frac{\partial \Psi}{\partial u_i} \dot{u}_i \right)\Bigg|_{t=t_0} = 0; \\ &\left(\frac{\partial^2 \Psi}{\partial u_i \partial u_j} \dot{u}_i \dot{u}_j + \frac{\partial \Psi}{\partial u_i} \ddot{u}_i \right)\Bigg|_{t=t_0} = 0 \Bigg\}, \end{aligned} \tag{4.155}$$

because, if we take into account the second derivative (acceleration) of the displacement field, then we must hold this derivative in the development (4.154). Notice once more that the acceleration for the initial moment is defined by the Gauss principle of minimum constraint [GPS02]. This result is due to Mayer (see Comments in [Ost61]).

We use two variational principles: the Lagrange principle of the virtual displacements and the Jourdain principle (see (6.3)). The Lagrange principle is appropriate to the solution of a quasi-static friction contact problem. For such a case we use the following definition of the admissible displacements:

$$K = \{v \mid v = v(x,t),\ x \in \Omega,\ t \in [0,T];\ v \in W(0,T);\\ v_N(x) \leq \delta_N(x),\ \forall x \in \Sigma_C^t\} \quad (4.156)$$

which is as before. The difference consists of the presence of an additional variable t describing an evolution of the contact interaction.

If we deal with a dynamic friction contact problem, we use the Jourdain principle. In such a case there are two unknown functions – a displacement field $u(x,t)$ and a velocity field $\dot{u}$. We can use the Jourdain principle in a quasi-static problem too, the number of unknown fields will be double with respect the Lagrange method.

Let $\dot{u}_i$ be the true velocity field, and let $\dot{u}_i + \delta\dot{u}_i$ be a kinematical admissible velocity field. We obtain the following constraint:

$$\delta\dot{u}_i \frac{\partial\Psi}{\partial u_i} \geq 0 \quad \forall x \in \Sigma_C^{t_0}. \quad (4.157)$$

The condition

$$\left.\left(\frac{\partial^2\Psi}{\partial u_i \partial u_j}\dot{u}_i\dot{u}_j + \frac{\partial\Psi}{\partial u_i}\ddot{u}_i\right)\right|_{t=t_0} = 0$$

vanishes in the definition of the set $\Sigma_C^{t_0}$ by (4.155) for quasi-static problems. Collecting the obtained results, we arrive at the following definition of the kinematical admissible velocity field $\dot{K}_u$:

$$\dot{K}_u = \{\dot{v} \mid \dot{v} = \dot{u} + \delta\dot{u};\ \nabla\Psi \cdot (\delta\dot{u}) \geq 0,\ \forall x \in \Sigma_C^t\}. \quad (4.158)$$

If the variable t changes then, in general, the set Σ_C^t and the function $\Psi = \Psi(x + u(x,t))$ change, too. It follows from the definition (4.158) that the set $\dot{K}_u$ depends on an unknown solution. Such a problem was analyzed first in [Lio75, Tar74]. The variational approach gives the so-called *quasi-variational inequalities.*

To obtain such an inequality for the friction contact problem, we return to the virtual velocity principle (4.146) and demonstrate the inequality

$$\int_{\Sigma_C} (\sigma_N \delta\dot{u}_N + \sigma_T \cdot \delta\dot{u}_T)\, d\Sigma \geq -\int_{\Sigma_C} f|\sigma_N(u)|(|\dot{v}_T| - |\dot{u}_T|)\, d\Sigma, \quad (4.159)$$

where $\dot{v}_T = \dot{u}_T + \delta\dot{u}_T$ is the kinematic admissible velocity field constrained by the inequality (4.157). Note that

$$\sigma_N \delta\dot{u}_N \geq 0. \tag{4.160}$$

Indeed, from (4.57) and (4.59) it follows that

$$\delta\dot{u}_i \frac{\partial \Psi}{\partial u_i} \approx -\delta\dot{u}_N \geq 0. \tag{4.161}$$

Taking into account the constraint (4.54) from (4.160), we obtain the inequality (4.159).

We now perform the transition from the local formulation to the variational one. By supposition, the Amonthon–Coulomb friction law (4.140) and (4.141) holds. The last equation in (4.141), written as

$$\sigma_T|\dot{u}_T| + |\sigma_T|\dot{u}_T = 0, \tag{4.162}$$

is valid for all the domain Σ_C, in the stick domain $\dot{u}_T = 0$, in the domain of the contact violation $\sigma_T = 0$, and in the slip domain

$$\frac{\sigma_T}{|\sigma_T|} = -\frac{\dot{u}_T}{|\dot{u}_T|}.$$

Suppose temporarily that the virtual velocity field $\dot{v}_T$ also satisfies the constraint (4.162) with same values of $\sigma_T = \sigma_T(u)$ as in (4.162). Subtraction gives

$$\sigma_T(|\dot{v}_T| - |\dot{u}_T|) + |\sigma_T|(\dot{v}_T - \dot{u}_T) = 0. \tag{4.163}$$

Then

$$\sigma_T \cdot \delta\dot{u}_T \equiv \sigma_T \cdot (\dot{v}_T - \dot{u}_T) = -|\sigma_T|(|\dot{v}_T| - |\dot{u}_T|). \tag{4.164}$$

The right-hand side of this equality is estimated from below by the value $-f|\sigma_N|(|\dot{v}_T| - |\dot{u}_T|)$. Indeed, if $|\sigma_T| = f|\sigma_N|$ then we get an equality. If $|\sigma_T| < f|\sigma_N|$ then $\dot{u}_T = 0$ and the right-hand side in (4.164) is nonnegative. Therefore,

$$\sigma_T \cdot (\dot{v}_T - \dot{u}_T) \geq -f|\sigma_N|(|\dot{v}_T| - |\dot{u}_T|). \tag{4.165}$$

We now show that the estimate (4.165) holds for a kinematic admissible state $\dot{v}$. To do this, analyze the sign of the expression

$$\dot{A} \equiv \sigma_T \cdot (\dot{v}_T - \dot{u}_T) + f|\sigma_N|(|\dot{v}_T| - |\dot{u}_T|). \tag{4.166}$$

Note, first, that from $|\sigma_T| < f|\sigma_N|$ it follows that $\dot{u}_T = 0$. Therefore, in this case

$$\dot{A} \equiv \sigma_T \cdot v_T + f|\sigma_N||\dot{v}_T| \geq \sigma_T \cdot \dot{v}_T + |\sigma_T||\dot{v}_T| \geq 0 \quad \forall \dot{v}.$$

If $|\sigma_T| = f|\sigma_N|$ then

$$\dot{A} = \sigma_T \cdot \dot{v}_T - \sigma_T \cdot \dot{u}_T + |\sigma_T||\dot{v}_T| - |\sigma_T||\dot{u}_T|. \quad (4.167)$$

From the condition (4.162) it follows that $\sigma_T \cdot \dot{u}_T = -|\sigma_T||\dot{u}_T|$. Therefore,

$$\dot{A} = |\sigma_T||\dot{v}_T|(1 - \cos(\dot{u}_T, \dot{v}_T)) \geq 0. \quad (4.168)$$

Q.E.D.

Thus, any solution of the boundary value problem of the linear theory of the elasticity (2.113)–(2.115), (2.229), (2.230), (4.53), (4.54), and (4.56) with the boundary condition (4.140) and (4.141) satisfies the inequality

$$a(u, \delta\dot{u}) - L(\delta\dot{u}) + \int_{\Sigma_C} f|\sigma_N(u)|(|\dot{v}_T| - |\dot{u}_T|)\, d\Sigma \geq -\int_\Omega \rho\ddot{u} \cdot \delta\dot{u}\, d\Omega$$
$$\forall \delta\dot{u} = \dot{v} - \dot{u},\ \dot{v} \in \dot{K}_u,\ \dot{u} \in \dot{K}_u,\ u \in K. \quad (4.169)$$

For quasi-static problems we have

$$a(u, \delta\dot{u}) + \int_{\Sigma_C} f|\sigma_N(u)|(|\dot{v}_T| - |\dot{u}_T|)\, d\Sigma \geq L(\delta\dot{u})$$
$$\forall \delta\dot{u} = \dot{v} - \dot{u},\ \dot{v} \in \dot{K}_u,\ \dot{u} \in \dot{K}_u,\ u \in K. \quad (4.170)$$

Using the terminology introduced by J.-L. Lions [Lio75] we call the inequality (4.169) or (4.170) a *quasi-variational inequality*. As it was pointed above, the main peculiarity of the quasi-variational inequality is the dependence of the admissible sets on the solution. This is a reason for creating special methods to solve the quasi-variational inequalities and investigate their properties, see, e.g., [Lio75]. These constraints can have the form $v \leq M(u)$, where $M(u)$ is, in general, a nonlinear operator.

Taking into account the boundary conditions classification (natural or forced) introduced earlier, we conclude that the constraint (4.143) is the natural condition, because it follows from the variational formulation.

4.3.3 Interpretation

Call the solution of the inequality (4.170) (or (4.169)) the *generalized solution* of the contact problem in the local formulation (2.113)–(2.115), (2.229), (2.230), (4.53), (4.54), (4.56), (4.140), and (4.141). We show that any twice differentiable solution of the inequality (4.169) satisfies all the local equations and conditions. The transition from the variational problem to the local formulation is called *interpretation*.

Repeating the demonstration performed in Section 4.2.2, we find that the solution of the variational inequality satisfies the motion equation

(2.113)–(2.115) and the boundary condition (2.230), (4.54). To obtain the conditions (4.140) and (4.141), note that these conditions are equivalent to the following relations:

$$\sigma_T|\dot{u}_T| + f|\sigma_N|\dot{u}_T = 0, \tag{4.171}$$

$$|\sigma_T| \leq f|\sigma_N|. \tag{4.172}$$

Indeed, if $|\sigma_T| \leq f|\sigma_N|$ then the only way to satisfy equation (4.171) is to choose $\dot{u}_T = 0$. If $|\sigma_T| = f|\sigma_N|$ then the equation (4.171) follows at once from equation (4.162). Q.E.D.

We show that the inequality (4.169) implies the conditions (4.171) and, consequently, the conditions (4.140) and (4.141). Take into account the fact that all the local relations are demonstrated except the conditions (4.140) and (4.141) and choose $\dot{u}_N = \dot{v}_N$ on Σ_C. Then from the inequality (4.169) we obtain

$$\int_{\Sigma_C} [\sigma_T \cdot (\dot{v}_T - \dot{u}_T) + f|\sigma_N|(|\dot{v}_T| - |\dot{u}_T|)]\, d\Sigma \geq 0. \tag{4.173}$$

Introduce the notation $\varphi(u, \dot{v}) \equiv \sigma_T \cdot \dot{v}_T + f|\sigma_N||\dot{v}_T|$. Then the inequality (4.173) takes the form

$$\int_{\Sigma_C} [\varphi(u, \dot{v}) - \varphi(u, \dot{u})]\, d\Sigma \geq 0. \tag{4.174}$$

Choose, first, $\dot{v}_T = 0$ and, secondly, $\dot{v}_T = 2\dot{u}_T$ (this choice does not violate the condition $v \in K \subset W$). We obtain the conclusion that

$$\int_{\Sigma_C} \varphi(u, \dot{u})\, d\Sigma = 0, \tag{4.175}$$

from which, using proof by contradiction, it follows that $\varphi(u, \dot{u}) = 0$, i.e., the condition (4.171).

To demonstrate the inequality (4.172), note that (4.174) and (4.175) give the estimate

$$\int_{\Sigma_C} f|\sigma_N||\dot{v}_T|\, d\Sigma \geq -\int_{\Sigma_C} \sigma_T \cdot \dot{v}_T\, d\Sigma. \tag{4.176}$$

Replace the quantity $\dot{v}_T$ by $-\dot{v}_T$ (this replacement does not violate the condition $v \in K \subset W$) and compare the obtained inequality with the inequality (4.176). We conclude that

$$\left| \int_{\Sigma_C} \sigma_T \cdot \dot{v}_T\, d\Sigma \right| \leq \int_{\Sigma_C} f|\sigma_N||\dot{v}_T|\, d\Sigma. \tag{4.177}$$

The following inequality is well known (analogous to the Cauchy inequality):

$$\left| \int_{\Sigma_C} \sigma_T \cdot \dot{v}_T\, d\Sigma \right| \leq \int_{\Sigma_C} |\sigma_T||\dot{v}_T|\, d\Sigma. \tag{4.178}$$

There is equality if and only if the direction of the vector σ_T coincides with the direction of the vector $\dot{v}_T$. Therefore, the "constant" $f|\sigma_N|$ in the estimate (4.177) is not less than the quantity $|\sigma_T|$ in the inequality (4.178). Q.E.D.

The results obtained in Sections 4.3.2 and 4.3.3 can be unified into the following theorem.

Theorem 4.7. *The contact friction problem of the linear elasticity with boundary conditions (2.229) on Σ_u and (2.230) on Σ_σ (see Chapter 2), impenetrability condition (4.53), nonpositivity of the contact pressure (4.54) together with (4.56), and the Coulomb friction law (4.140) and (4.141) is equivalent to the quasi-variational inequality (4.169).*

Note that this inequality is appropriate for theoretical investigations. To solve a quasi-static friction contact problem, in the next section we propose a method related to the Lagrange principle and the equation (4.146).

4.3.4 Solution of the quasi-variational inequality

An idea of the method for solution of the friction contact problem was found in saddle problems researches [AHU58]. It is related to the initial variational equation (4.146). A solution must satisfy the constraints (4.53), (4.54), (4.56), (4.140)–(4.143). Then the problem is nonlinear due to the constraints which are inequalities. Note that the proof of convergence is an open problem. A test solution shows that the proposed iteration process converges.

A solution of a friction contact problem depends, in general, on the history of evolution of an external forces. The method relates to the problems where we find the whole process of the system evolution. Such a problem is reduced to the finding of an unknown field increments corresponding to the prescribed increments of external forces.

To emphasize the difference between the scalars and vectors, in this section vectors are printed in bold type. Let the field solution $u(x,t)$ and the increment

$$d\mathbf{u}(\mathbf{x},t) = \dot{\mathbf{u}}(\mathbf{x},t)dt \simeq \mathbf{u}(\mathbf{x},t+dt) - \mathbf{u}(\mathbf{x},t) \equiv \mathbf{u} - \mathbf{u}^t \tag{4.179}$$

be unknown.

We now consider a quasi-static friction contact problem and, using the definition (4.179), we write the equation (4.146) as follows:

$$a(\mathbf{u}^{t+dt}, \delta\mathbf{u}) = L^{t+dt}(\delta\mathbf{u}) + \int_{\Sigma_C} (\sigma_N \delta u_N + \boldsymbol{\sigma}_T \cdot \delta\mathbf{u}_T)\, d\Sigma, \tag{4.180}$$

where $\delta\mathbf{u} = \mathbf{v} - \mathbf{u}^{t+dt}$. The functions $\mathbf{u}$, σ_N, $\boldsymbol{\sigma}_T$ satisfy (4.53), (4.54), (4.56), (4.140)–(4.143).

The following iterative process (r is the iteration number) is proposed:

Step 1. Choose in the equation (4.180) a zero-approximation $\sigma_N^{(0)}$, $\boldsymbol{\sigma}_T^{(0)}$ for the contact stresses.

Step 2. Find the solution of this equation with any restriction on δu. Such the problem is equivalent to the BVPs of linear elasticity with the boundary conditions on the surface Σ_C

$$\sigma_{ij}\nu_j|_{\Sigma_C} = \sigma_N^{(0)}\nu_i + (\boldsymbol{\sigma}_T^{(0)})_i \tag{4.181}$$

and with the old boundary condition on Σ_u, Σ_σ. As a result we have the displacement field $\mathbf{u}^{(0)}$.

Step 3. To satisfy all the constrains (4.53), (4.54), (4.56), (4.140)–(4.143), we modify the contact stresses by

$$\sigma_N^{(1)} = P_N(\sigma_N^{(0)} + \rho_{0N}\Psi(\mathbf{x} + \mathbf{u}^{(0)}(\mathbf{x}))),\ \mathbf{x} \in \Sigma_C, \tag{4.182}$$

$$\boldsymbol{\sigma}_T^{(1)} = P_T(\boldsymbol{\sigma}_T^{(0)} + \rho_{0T}(\mathbf{u}_T^{(0)} - \mathbf{u}_T^t)), \tag{4.183}$$

where

$$P_N(\sigma_N) = \begin{cases} \sigma_N, & \sigma_N \le 0, \\ 0, & \sigma_N > 0, \end{cases}$$

$$P_T(\boldsymbol{\sigma}_T) = \begin{cases} \boldsymbol{\sigma}_T, & |\boldsymbol{\sigma}_T| \le f|\sigma_N^{(0)}|, \\ \dfrac{\boldsymbol{\sigma}_T}{|\boldsymbol{\sigma}_T|} f|\sigma_N^{(0)}|, & |\boldsymbol{\sigma}_T| > f|\sigma_N^{(0)}| \end{cases}$$

are the orthogonal projection operators of the corrected contact stresses on the admissible set of contact stresses defined by the condition of nonpositivity of the normal pressure and the Coulomb friction law. ρ_{0N} and ρ_{0T} are the numerical parameters controlling the convergence rate.

Step 4. If the stop criterium is not satisfied, we go to Step 2 with the new values of the contact stresses.

This time the described procedure is a heuristic method in which the correction of normal pressure is proportional to an error in impenetrability requirement, and correction of the friction forces is proportional to the current value of the relative slip velocity. A partial foundation of the proposed method will be given in Chapter 5.

4.3.5 A priori estimate of the dynamic problem solution

We deduce this estimate using the norms introduced earlier and establish the hypothesis (4.151). To do this, choose $\dot{v} = 0$ in equation (4.146). This choice is admissible because this substitution leads to an equation which formally can be deduced by scalar multiplication of the initial equation by the quantity $\dot{u}$, integration over the whole domain and the application of the Green formula. Taking into account the equality $\sigma_N \dot{u}_N = 0$ and the fact that the vectors $\dot{u}_T$, σ_T are antiparallel, we rewrite the result in the following form:

$$\frac{d}{dt}\left[\int_{\Omega}\frac{\rho}{2}|\dot{u}|^2\,d\Omega+\frac{1}{2}a(u,u)\right]+\int_{\Sigma_C}|\sigma_T(u)||\dot{u}_T|\,d\Sigma=L(\dot{u}). \tag{4.184}$$

Integrate the equality (4.184) on τ from 0 to t:

$$\int_{\Omega}\frac{\rho}{2}|\dot{u}|^2\,d\Omega+\frac{1}{2}a(u,u)+\int_0^t\int_{\Sigma_C}|\sigma_T(u)||\dot{u}_T|\,d\Sigma\,d\tau$$
$$=\int_{\Omega}\frac{\rho}{2}|\dot{u}(0)|^2\,d\Omega+\frac{1}{2}a(u(0),u(0))+\int_0^t L(\dot{u}(0))\,d\tau. \tag{4.185}$$

Use the well-known inequality [VH85]

$$\int_0^t L(\dot{u}(\tau))\,d\tau\le\frac{1}{2\varepsilon}\int_0^t\|L\|^2\,d\tau+\frac{\varepsilon}{2}\int_0^t\int_{\Omega}|\dot{u}|^2\,d\Omega\,d\tau, \tag{4.186}$$

where $\|L\|$ is the norm of the linear form L. Use the positive definiteness inequality

$$a(v,v)\ge\alpha\|v\|^2,\quad \alpha=\text{const}>0,$$

choose $\min\{\rho/2,\alpha/2\}=c_0$, $\varepsilon/2=c_0$, and reject the nonnegative quantity

$$\int_0^t\int_{\Sigma_C}|\sigma_T||\dot{u}_T|\,d\Sigma\,d\tau$$

on the left-hand side of the equality (4.185). Add the integral

$$\frac{c_0}{2}\int_0^t\|u(\tau)\|_V^2\,d\tau$$

to the right-hand side of (4.185).

The result of all these operations will be the inequality

$$c_0\varphi(t)\le c_1+\frac{1}{2\varepsilon}\int_0^t\|L\|^2\,d\tau+c_0\int_0^t\varphi(\tau)\,d\tau, \tag{4.187}$$

where

$$\varphi(t)=\int_{\Omega}|\dot{u}|^2\,d\Omega+\|u\|_V^2,\quad c_1=\int_{\Omega}\frac{\rho}{2}|\dot{u}(0)|^2\,d\Omega+\frac{1}{2}a(u(0),u(0)).$$

Application of the Gronwall inequality [BS92] in (4.187) gives the final result

$$\varphi(t)\le\left(\frac{1}{4c_0^2}\int_0^t\|L\|\,d\tau+c_1\right)\exp(t), \tag{4.188}$$

from which we obtain the following conclusion:

$$u(t)\in L^2(0,T;V),\quad \dot{u}(t)\in L^2(0,T;V). \tag{4.189}$$

Q.E.D.

4.3.6 Dynamic contact of an elastic solid and a system of rigid moving indenters

Results of this section were published in [KN03b].

Formulation of the problem

Let us consider the problem of the motion of a deformable solid Ω. $\partial\Omega = \Sigma$, the boundary of Ω, is in contact with a system of rigid moving indenters. The problem is to determine the displacements, stresses, strains, the contact domain, and the contact pressures acting on the body Ω. An appropriate mathematical model consists of the motion equation for the solid Ω and all the indenters with the corresponding boundary and initial conditions.

The deformable solid is linearly elastic with the modulus of elasticity tensor ${}^4\hat{a}$. Denote by $u = u(x,t)$ the displacement field, $\hat{\varepsilon} = \hat{\varepsilon}(u)$ the strain tensor and $\hat{\sigma}$ the stress tensor. The motion equation for the solid Ω is the following:

$$\nabla \cdot ({}^4\hat{a} \cdot \cdot \hat{\varepsilon}(u)) = \rho \frac{\partial^2 u}{\partial t^2}, \tag{4.190}$$

where ρ is the material density, ∇ is the Hamilton operator and the symbol "$\cdot$" denotes the scalar product.

Suppose that the boundary of the deformable solid is the union of the surfaces

$$\Sigma = \Sigma_u \cup \Sigma_\sigma \cup \Sigma_C^1 \ldots \cup \Sigma_C^{L_{\max}},$$

where Σ_u is the fixed part of the boundary, and the surface tractions $P(x,t)$ are prescribed on Σ_σ:

$$\hat{\sigma} \cdot \nu|_{\Sigma_\sigma} = P(x,t), \tag{4.191}$$

where ν is the unit external vector orthogonal to the surface Σ.

The boundary conditions on the parts Σ_C^L, $L = 1, 2, \ldots, L_{\max}$, describe the impenetrability requirement for the indenter number "L", the nonnegativity of the normal pressure $\sigma_N = [\hat{\sigma} \cdot \nu] \cdot \nu$, and the equality $\sigma_T = \hat{\sigma} \cdot \nu - (\sigma_N)\nu = 0$ which means that there is no friction. The last hypothesis can be overcome with the results of earlier sections.

To formulate the impenetrability requirement, suppose that the boundary of the indenter number "L" for $t = 0$ is given by the equation

$$\Psi^L(x) = 0 \tag{4.192}$$

with respect to the fixed coordinate system $O(x_1 x_2 x_3)$.

Let $O_1(\xi_1 \xi_2 \xi_3)$ be the moving coordinate system rigidly attached to the moving indenter which coincides with the fixed system $O(x_1 x_2 x_3)$ for $(t = 0)$. Then the equations for the indenter boundary points for all values of the time variable have the form (4.192). The dependence of the variable $x(t)$ on the vector ξ corresponding to the same material point is given by the equation

$$x(t) = U_p + [A]\xi, \tag{4.193}$$

where the vector U_p describes the translation motion of the indenter, and the matrix $[A]$ depends on the Euler angles corresponding to the body rotation. Generally speaking, the equation (4.193) is valid for any infinitesimal quantities.

Choose the function $\Psi^L(x)$ to be negative inside the indenter and positive outside. With this hypothesis the impenetrability requirement will be the following:

$$\Psi^L([A]^{-1}(x - U_p + u(x))) \geq 0, \quad \forall x \in \Sigma_C^L. \tag{4.194}$$

The local formulation of the problem (local mathematical model) is to find the displacement field $u(x,t)$, $x \in \Omega$, $t \in [0, T]$, and corresponding strain and stress tensors from the equations (4.190), boundary conditions on Σ_u, Σ_σ, non-negativity normal efforts $\sigma_N = [\hat{\sigma} \cdot \nu] \cdot \nu \leq 0$ requirement, friction absence $\sigma_T = \hat{\sigma} \cdot \nu - (\sigma_N)\nu = 0$ and impenetrability requirement (4.194). The motion equation for each indenter must be used, too. It is well known that there are six such equations. They include the unknown contact stresses and kinematic parameters with prescribed resultant and moment for each indenter or contact stresses, resultant and moment for prescribed movement of each indenter.

Variational approach

The variational formulation of the contact problem follows from the variational principle of admissible velocities. The essential problem is to find the admissible velocity set. The leading concept given by M. V. Ostrogradski consists in taking into account only the constraints at the points of the boundary Σ_C^t which are in contact with the indenter. Their velocities and accelerations are equal to zero with respect to the indenter. According to the Mayer and Zermelo theorem (see [Ost61, Comments]), the initial accelerations are defined by the Gauss minimal principle.

The set Σ_C^t depends on the solution $u(x,t)$. Thus, the admissible velocity set $\dot{K}_u$ depends on the solution, too. Using these arguments, we can show that the variety Σ_C^t is defined as follows (the index, indenter number is omitted to simplify the notation):

$$\Sigma_C^t = \Big\{ x \mid x \in \Sigma_C;\ \Psi(\alpha(x)) = 0;$$
$$\frac{\partial \Psi(\alpha(x))}{\partial \alpha} \cdot ([\dot{A}]^{-1} \cdot (x - U_p + u) + [A]' \cdot ((-U_p + u)'_t)) = 0;$$
$$\Psi(\alpha)''_{\alpha\alpha} \cdot \cdot (\alpha'_t \otimes \alpha'_t) + \frac{\partial \Psi(\alpha(x))}{\partial \alpha} \cdot \alpha''_{tt} = 0 \Big\}, \tag{4.195}$$

where "$\cdot$" denotes the derivative with respect to time, "$'$" denotes the derivative with respect to the variable given as the subscript, $\alpha = [A]^{-1} \cdot (x - U_p + u)$, and $\bigotimes$ is dyadic multiplication.

According to the Ostrogradski principle, the admissible velocity set is defined as follows:

$$\dot{K}_u = \{v'_t \mid v'_t = u'_t + \delta u'_t; \Psi(\alpha)'_\alpha \cdot (A^{-1} \cdot \delta u'_t) \geq 0,\ \forall x \in \Sigma^t_C\}, \qquad (4.196)$$

where u is the solution of the problem. If the indenter is fixed then the matrix A is identity and the formulae (4.195) and (4.196) imply the impenetrability condition for a fixed stamp.

Using the definition (4.196), we can construct the corresponding quasi-variational inequality. This inequality can be solved by means of step-by-step methods in time with the simultaneous iteration procedures given earlier to find the contact domains, slip and stick domains for contact problems with friction for the contact of a fixed stamp and an elastic body.

4.3.7 Local potential method

To solve the problem like to the inequality (4.170) with two unknown fields, I. Prigogine proposed a so-called *local potential method.* First this method was given for chemical kinetic problems. The local potential method for the quasi-static problems consists of finding two functions $u(x, t + dt) \equiv u$ and v minimizing the functional

$$J(u, v) = \frac{1}{2}a(v, v) - L(v) + \int_{\Sigma_C} f|\sigma_N(u)|(|v_T| - |u_T|)\, d\Sigma, \qquad (4.197)$$

where at the minimum point the two varied functions must coincide. Therefore, the question is to solve the conditional minimization problem in which the two varied functions must satisfy the constraint, being equal at the minimum. In practice, the solution algorithm is realized in the following way. It is supposed temporarily that the element u in the functional (4.197) is known, and the minimization problem is solved without constraints

$$J(u, v) \to \min_{v \in K}. \qquad (4.198)$$

Denote the solution of the problem (4.198) by $\tilde{u}$. We must choose from the set $\{\tilde{u}\}$ of all the solutions corresponding to the different elements u to find the solution $\tilde{u}_0$ which is equal to the element u.

Notice that there is no effective solution method of the formulated problem mainly because of the large number of unknown functions.

4.3.8 Proportional processes

It can be seen that the friction law (4.140) and (4.141) leads to problems where we must take into account all the history of the stress and strain evolution corresponding to the history of the external actions. In practice, and in some theoretical investigations, the contact problems are solved just at once for the final state of the external actions, with substitution of the velocities in the governing friction law (4.140) and (4.141) by the displacements. It was found that this method often gives good results. If it takes into account the homogeneity of the relation (4.141) with respect to time (i.e., the absolute value of the slide velocity does not influence the evolution processes) then the substitution of the velocities by the displacements is equivalent to the use of a single step in the step method of the system evolution investigation.

This situation is analogous to that which takes place in the deformation theory of plasticity. It was shown by A. A. Ilyushin for a class of deformation processes called simple (or proportional) processes when the rate deformation law of the plasticity theory is replaced by the governing law with deformation only. This hypothesis can be established for some special cases of loading [Kra01]. An analogous foundation can be given for the Amonthon–Coulomb friction law.

We formulate this result as the following theorem.

Theorem 4.8. *If the direction field of the vector σ_T on Σ_C does not change with the change of the external actions and its absolute value is changed proportional to a load parameter λ, i.e.,*

$$\sigma_T = \sigma_T^0 \lambda, \quad \sigma_T^0 = \sigma_T^0(x), \ x \in \Sigma_C, \ d\lambda \geq 0, \tag{4.199}$$

then the Amonthon–Coulomb relations (4.140)–(4.143) are valid for the displacements.

Proof. It can be seen from the relation (4.141) and the formula (4.109) that at any point $x \in \Sigma_C$ we have

$$u_T = -\left(\int_0^{|u_T|} |du_T|\right) \frac{\sigma_T}{|\sigma_T|} \equiv k\sigma_T, \tag{4.200}$$

i.e., the tangent displacement vector u_T is parallel to the vector σ_T. By supposition, the direction field of the vector σ_T is constant, so that the direction field of the vector u_T is constant, too. Therefore,

$$|du_T| = d|u_T|, \tag{4.201}$$

and if $|u_T| \neq 0$ then from the relation (4.200) we get

$$\frac{u_T}{|u_T|} = -\frac{\sigma_T}{|\sigma_T|}, \quad |\sigma_T| = f|\sigma_N|. \tag{4.202}$$

Q.E.D.

Notice that the supposition (4.199) must be valid at any point x in which the sliding exists. The zero value in the formula (4.199) corresponds to this moment in time. If there is no sliding then

$$|\sigma_T| < f|\sigma_N|, \quad u_T = 0. \tag{4.203}$$

This relation coincides with the second relation in the Amonthon–Coulomb friction law. The formulae (4.200)–(4.203) are deduced from the hypothesis on the "activity" of the loading process at the current point. This supposition means that after sliding begins at the given point $x \in \Sigma_C$ the sliding velocity must not be zero. In principle, this constraint can be eliminated. Indeed, if at some moment the quantity λ begins to decrease, then the formula (4.200) must be applied at first to the time segment with the monotone increasing displacements. After that instead of the formula (4.200) we must use another formula where the displacement u_T corresponds to a state with a monotonic decrease of the parameter λ.

The validity of the formulae (4.200) and (4.201) is an open question. A special case for which such a solution exists is found by the following statement. If contact, sliding and adhesion are constant with the evolution of the system loads then the conditions holds if the external actions are changed proportionally to a single common parameter. Demonstration of this statement is performed by substitution of the enumerated quantities in the conditions (4.200).

4.3.9 Further generalization

Generalized friction law

The Amonthon–Coulomb friction law (4.140)–(4.143) sometimes gives absurd results, e.g., it leads to infinite values of the tangent contact stress without plastic flow of material. More realistic from the physical point of view is the following friction law:

$$\begin{cases} \text{If } |\sigma_T| < f|\sigma_N| \text{ and simultaneously } |\sigma_T| < \tau_s \text{ then } \dot{u}_T = 0; \\ \text{If } |\sigma_T| = f|\sigma_N| \text{ or } |\sigma_T| = \tau_s \text{ then } \exists \kappa \geq 0 : \dot{u}_T = -\kappa \sigma_T. \end{cases} \tag{4.204}$$

This law is used, in particular, in the works of A. A. Ilyushin and S. S. Grigoryan. The constant τ_s in (4.140) is the shearing flow limit which can depend on the deformations and deformation rates. f is the friction coefficient depending, in general, on the relative velocity of sliding and other characteristics of the contact.

A method, analogous to that used to deduce the inequalities (4.169) and (4.170), allows us to demonstrate the following statement. The problem of defining the stresses and strains of the elastic body Ω loaded by the volume forces F, surface tractions P on the part Σ_σ of the boundary, and with clamped part Σ_u for the friction law (4.125) is equivalent (in quasi-statics) to the variational inequality

$$\begin{aligned} a(u, v-u) - L(v-u) + \int_{\Sigma_C} [f|\sigma_N| + (\tau_s - f|\sigma_N|) \\ \times \theta_0(f|\sigma_N| - \tau_s)][|v - u^t| - |u - u^t|]\, d\Sigma \geq 0 \quad \forall v \in K,\ u \in K, \end{aligned} \tag{4.205}$$

where θ_0 is the Heaviside step function.

Contact problem for a number of deformed bodies

Suppose that the deformations are small. To solve the problem, we must define the relative velocity of the surface points of one body with respect to another

$$\dot{u}_T^{\alpha\beta} = \dot{u}_T^{\alpha} - \dot{u}_T^{\beta}. \tag{4.206}$$

Using the variational equation (4.97) and estimates similar to them obtained in Section 4.3.2, we demonstrate that the dynamic problem for the contact of many bodies with dry friction is equivalent to the variational inequality

$$\sum_{\alpha} \int_{\Omega^\alpha} \frac{\partial^2 u^\alpha}{\partial t^2} \cdot \delta\dot{u}^\alpha \, d\Omega + a(u, \delta\dot{u}) + \sum_{\alpha} \int_{\Sigma_C^\alpha} f^{\alpha\beta} |\sigma_N^{\alpha\beta}| (|\dot{v}_T^{\alpha\beta}| - |\dot{u}_T^{\alpha\beta}|) \, d\Sigma$$
$$\geq L(\delta\dot{u}) \quad \forall \delta\dot{u} = \dot{v} - \dot{u}; \ \dot{v} \in \dot{K}_u, \ \dot{u} \in \dot{K}_u, \ u \in K, \tag{4.207}$$

where the functionals $a(u, \delta\dot{u})$ and $L(\delta\dot{u})$ are defined by the formulae (4.94) and (4.95), index α is to the body number, and $f^{\alpha\beta}$ is the friction coefficient for the contacting bodies α and β. In the definition of the set K we use one of the forms of the impenetrability condition proposed earlier.

Quasi-static problems are equivalent to the following inequality:

$$a(u, v-u) - L(v-u) + \sum_{\alpha} \int_{\Sigma_C^\alpha} f^{\alpha\beta} |\sigma_N(u^\alpha, u^\beta)| (|v_T^{\alpha\beta} - u_T^{t\alpha\beta}|$$
$$- |u_T^{\alpha\beta} - u_T^{t\alpha\beta}|) \, d\Sigma \geq 0 \quad \forall v \in K, \ u \in K, \tag{4.208}$$

where the function $u_T^{t\alpha\beta}$ is known from the previous step of a loading process.

If we consider a contact problem without a fixed part of the boundary then it is impossible to find *a priori* the friction force power estimations of the type (4.160), which do not depend on the external loads. Thus, it is impossible to deduce the conditions of the type (4.121) and (4.123) providing the stability of the contacting bodies as a whole. We can point out only a limit inequality that must be valid for each loading step. This inequality follows from (4.208) with the substitution $v - u = y$ where y is the rigid displacement set from the equilibrium state. For such variations, under the impenetrability condition, we get $a(u, y) = 0$. Therefore,

$$L(y) \leq \sum_{\alpha} \int_{\Sigma_C^\alpha} f |\sigma_N| (|y_T + u_T - u_T^t| - |u_T - u_T^t|) \, d\Sigma. \tag{4.209}$$

For simplicity the indices α and β are omitted. If we exclude the neutral equilibrium states, then the inequality (4.209) must be strong for zero element y (in correspondence with the Signorini strong hypothesis).

Contact problem with friction for large displacements and strains

Consider the quasi-static friction contact problem investigated in Section 4.2.4. Suppose that instead of the condition $t_T = 0$ (see the formula (4.130)) the friction law of Amonthon–Coulomb (4.140)–(4.143) holds with the substitutions σ_N for τ_N and σ_T for τ_T. Using the Lagrange variable and the virtual velocity principle, we can demonstrate that the problem (4.125)–(4.131) is equivalent to the variational inequality

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta\dot{u}_{j,i}\, d\Omega_0 = \int_{\Omega_0} \rho_0 F_0 \cdot \delta\dot{u}\, d\Omega_0 + \int_{\Sigma_{0\sigma}} kP \cdot \delta\dot{u}\, d\Sigma_0 + \int_{\Sigma_{0c}} t_0^{(\nu)} \cdot \delta\dot{u}\, d\Sigma_0 \quad \forall \delta\dot{u},$$
$$\delta\dot{u} = \dot{v} - \dot{u}, \tag{4.210}$$

where $\delta\dot{u}$ is the virtual velocity under constraint

$$\delta\dot{u}(a, t + dt) \cdot \nabla\Psi(a + u(a, t + dt) \geq 0 \quad \forall a \in \Sigma_{0C}^t \tag{4.211}$$

with

$$\Sigma_{0C}^t = \{a \mid a \in \Sigma_{0c};\ \Psi(a + u(a,t)) = 0;\ \dot{u}(a,t) \cdot \nabla\Psi(a + u(a,t)) = 0\}\,. \tag{4.212}$$

Using the decomposition (compare with the formula (4.137))

$$t_0^{(\nu)} \cdot \delta u = k(u)[t_N \delta\dot{u}_N + t_T \cdot \delta\dot{u}_T], \tag{4.213}$$

the estimate for the quantity $t_N \delta\dot{u}_N$

$$t_N \delta\dot{u}_N \geq 0 \tag{4.214}$$

(which can be deduced to the estimate (4.160)) and the estimate

$$t_T \cdot \delta u_T \geq -f|t_N(u)|(|\dot{v}_T| - |\dot{u}_T|), \tag{4.215}$$

analogous to the inequality (4.165), we conclude that the problem is equivalent to the inequality

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)\delta\dot{u}_{j,i}\, d\Omega_0 + \int_{\Sigma_{0C}} f|t_N(u)|(|\dot{v}_T| - |\dot{u}_T|)\, d\Sigma_0$$
$$\geq L(u, \delta\dot{u}) \equiv \int_{\Omega_0} \rho_0 F_0 \cdot \delta\dot{u}\, d\Omega_0 + \int_{\Sigma_{0\sigma}} kP \cdot \delta\dot{u}\, d\Sigma_0, \tag{4.216}$$

where the quantity $\delta\dot{u} = \dot{v} - \dot{u}$ satisfies the constraint (4.211).

Using the formula (4.179), we can transform the inequality (4.216) to the following form:

$$\int_{\Omega_0} s_0^{ir}(\delta_r^j + u_{,r}^j)(v_{j,i} - u_{j,i})\, d\Omega_0$$
$$+ \int_{\Sigma_{0C}} f|t_N(u)|(|v_T - u_T^t| - |v_T - u_T^t|)\, d\Sigma_0 \geq L(u, v - u)$$
$$\forall \delta u = v - u \in K_u,\ v \in K,\ u \in K, \qquad (4.217)$$

where the sets K and K_u are defined by the formulae (4.134) and (4.135). The inequality (4.217) belongs to the quasi-variational type – this statement follows from the definitions given above. Investigation of this inequality is complicated by the possible nonpotentiality of the operator $A(u)$ introduced by the formula (3.232).

4.3.10 Comments

Apparently the first work on the variational approach to the contact problem with friction in the strict mathematical formulation was that by G. Duvaut and J.-L. Lions published in 1971. Exhaustive presentation of the results with some generalization is given in [DL72]. One of the essential hypotheses of these works was that the normal contact pressure was supposed to be known. This hypothesis was eliminated in [Kra80]. Note that, in many important technical problems, the mutual influence of the normal and tangent components of the contact force is small. Therefore, it is possible to first find the normal pressure σ_N on the supposition that friction is absent. After that we must find the tangential component σ_T with the prescribed normal component σ_N. There are situations where the problems on the determination of the tractions σ_N and σ_T separate strongly. Such a situation occurs for contact problems in the Hertz formulation where the contacting bodies are replaced by half-spaces with the identical Poisson ratios. The mutual influence of the tangential and normal contact stresses is taken into account in [Kra80] (see also Chapter 7). The fact that this influence is small is detected in [KS81], too. Another hypothesis of the theory developed by G. Duvaut and J.-L. Lions was the possibility of using displacements instead of velocities in the Amonthon–Coulomb friction law. The conditions for the validity of such hypothesis can be found in [Kra01] (see Section 4.3.8). A corresponding generalization of the relations (4.140) and (4.141) is given in [Kra80].

An original approach to the solution of the contact problems with the friction law (4.202) and (4.203)) is developed in [VS81]. The authors use (as do G. Duvaut and J.-L. Lions) the hypothesis that the normal pressure σ_N is known. This supposition allows us to prove that the contact problem reduces to the minimization of the functional (see the monograph [DL72])

$$J(v) = \frac{1}{2}a(v, v) - L(v) + \int_{\Sigma_C} g|v_T|\, d\Sigma, \qquad (4.218)$$

where $g = f|\sigma_N|$ is a known quantity. Admissible functions v satisfy the impenetrability condition. The first step of the method developed in [VS81] consists in using, instead of the last term in (4.218), the integral

$$\int_{\Sigma_C} gy \, d\Sigma$$

with the simultaneous introduction of the constraint

$$y = |v_T| \quad \text{on } \Sigma_C \tag{4.219}$$

on the additional unknown function y. The second step is the transition from the constraint (4.219) to the constraint

$$y \geq |v_T|, \tag{4.220}$$

which is convex, unlike the constraint (4.220). This property permits us to use theorems on convex functionals defined on convex sets. Another method for constructing the given algorithm will be described in Chapter 5.

The third and last step is the interpretation of the restriction (4.220) as the action of an ideal unilateral bond for the contacting bodies having a surface microrelief (irregularities of the boundary). Such an interpretation allows us to use the methods developed earlier for contact problems without friction and for contact problems with friction. This opens the way for the solutions of some hard contact problems.

In conclusion, note that the friction phenomenon in bearing design leads to important but difficult problems. The first strong mathematical investigation of this problem was performed by J. Kalker [Kal66]. The rolling of one body on the boundary of another body reduces to a minimization problem under constraint [Spe77]:

$$\min_{\substack{|\sigma_T| \leq f|\sigma_N| \\ s = B(\sigma_T) + v}} \left\{ F(\sigma_T, s(\sigma_T)) = \int_{\Sigma_C} [-f\sigma_N |s(\sigma_T)| - \sigma_T \cdot s] \, d\Sigma \right\}, \tag{4.221}$$

where s is the sliding velocity equal to the sum of the slip velocity v of the contacting bodies, supposed rigid, and the term $B(\sigma_T)$ defined by the deformation. The form of the operator B is given in [Spe77].

5

Transformation of Variational Principles

5.1 Friedrichs transformation

5.1.1 Introduction

In this section we focus on the problem of transforming variational principles so that one may exclude some of the variables in the general system of equations (such systems as (2.113)–(2.117)) and exchange roles between natural and forced conditions.

First, we consider the problem described by R. Courant and D. Hilbert [CH53]: find the minimum value of the function $f(x, y)$ on a curve given by the equation $g(x, y) = 0$. Let this problem be defined as follows:

Problem 5.1.

$$\begin{cases} f(x,y) \longrightarrow \min, \\ g(x,y) = 0. \end{cases} \tag{5.1}$$

Using the method of Lagrange multipliers, we construct the Lagrange function $\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda g(x, y)$ and consider the problem:

Problem 5.2.

$$\mathcal{L}(x,y;\lambda) = f(x,y) + \lambda g(x,y) \longrightarrow \underset{x,y,\lambda}{\text{stat}} \tag{5.2}$$

which investigates the stationary point of the function of three variables $(x, y; \lambda)$ without any constraints for the variables x, y, λ.

Assuming that the functions f and g are differentiable and using the stationary state condition of the function $\mathcal{L}$ by setting the partial derivatives of this function to zero, we formulate the problem:

Problem 5.3.

$$\begin{cases} \mathcal{L}'_x = 0 & \Longrightarrow \quad f'_x + \lambda g'_x = 0, \\ \mathcal{L}'_y = 0 & \Longrightarrow \quad f'_y + \lambda g'_y = 0, \\ \mathcal{L}'_\lambda = 0 & \Longrightarrow \quad g(x,y) = 0. \end{cases} \tag{5.3}$$

This is a system of three equations in three unknowns $\tilde{x}$, $\tilde{y}$, $\tilde{\lambda}$. Throughout the book, the symbol "~" will refer to the stationary point.

We apply now the statement given in [CH53]. If a function has a stationary value at some point, and if its arguments satisfy some equation at this stationary point, then the stationary point search problem for this function under the constraint, which is the equation at the stationary point, will be the same as for the initial stationary point problem without any additional constraint. For brevity, we will call this statement the "Hilbert principle."

Let us apply this principle to Problems 5.2 and 5.3: if we add the last of the equation (5.3) as a constraint to Problem 5.2, we will obtain Problem 5.1. If we add the first two of the equations (5.3) as constraints to Problem 5.2, we will make the *Friedrichs transformation* or *duality transformation*, see the definition in [CH53, Chapter IV, Section 1, p. 164]. The essence of this transformation is the elimination of the "old" variables (x, y) in the intermediate Problem 5.2. To execute the elimination, we compute (x, y) in the equations (5.3) as functions of the parameter λ:

$$x = x(\lambda), \quad y = y(\lambda) \tag{5.4}$$

and then substitute the expressions (5.4) into (5.2):

Problem 5.4.

$$\mathcal{L}(x(\lambda), y(\lambda); \lambda) \equiv f^*(\lambda) \longrightarrow \underset{\lambda}{\text{stat}} \tag{5.5}$$

putting no constraints on λ.

We now find the relation between the functions $d = f(\tilde{x}, \tilde{y})$ and $d^* = f^*(\tilde{\lambda})$ at the stationary points $(\tilde{x}, \tilde{y}; \tilde{\lambda})$. For this purpose, we define the number set d_λ according to the formula

$$d_\lambda = \min_{x,y} \mathcal{L}(x, y; \lambda). \tag{5.6}$$

As extending the domain of a function does not increase the minimum value of the function, then

$$d_\lambda \leq d. \tag{5.7}$$

(d is the minimum of the function $\mathcal{L}(x, y; \lambda)$ on the curve $g(x, y) = 0$. d_λ is the minimum of the same function on the whole plane (x, y).)

If, as on the curve $g(x, y) = 0$, the inequality (5.7) is transformed into an equality then

$$d = \max_\lambda d_\lambda = \max_\lambda \min_{x,y} \mathcal{L}(x, y; \lambda) = \max_\lambda f^*(\lambda) = d^*. \tag{5.8}$$

This series of equalities will repeatedly be used throughout this book. It demonstrates that if the original problem, Problem 5.1, is a minimization problem, then the intermediate problem, Problem 5.2 for the Lagrange function $\mathcal{L}(x, y; \lambda)$, is a problem of computing a saddle point (minimum on the old variables (x, y) and maximum on the Lagrange multiplier λ). If Problem 5.1 is a maximization problem then the transformed problem will be a minimization.

5.1.2 Boundary value problem for the ordinary differential equation

Let us summarize and tailor the ideas discussed in Section 5.1.1 for a boundary value problem (see Section 2.2):

$$-\frac{d}{dx}\left[ES(x)\frac{du}{dx}\right] = F(x), \quad 0 < x < l, \tag{5.9}$$

$$u(0) = u_0, \tag{5.10}$$

$$u(l) = u_l. \tag{5.11}$$

First, we transform the problems (5.9)–(5.11) into a minimization problem of a functional:

$$J(v) = \frac{1}{2}\int_0^l ES(x)\left(\frac{dv}{dx}\right)^2 dx - \int_0^l Fv dx \tag{5.12}$$

on a set of functions, satisfying the condition

$$v(0) = u_0, \quad v(l) = u_l. \tag{5.13}$$

This is Problem 5.1. We notice that this transformation is valid if the conditions (2.49) are satisfied. Moreover, $v \in H^1(0, l)$ (see Section 2.2). Thus, the constraint (5.13) is the only constraint. However, this "constraint at a point" is insufficient to obtain any interesting results.

The basic approach, which allows meaningful results to be obtained, involves the idea of declaring the function v and its derivative $v' = dv/dx$ to be independent arguments of the functional (5.12). As a result, we formulate the following problem:

Problem 5.5.

$$J(v, v') = \frac{1}{2}\int_0^l ES(x)(v')^2 dx - \int_0^l Fv dx \longrightarrow \min_{v,v'}, \tag{5.14}$$

$$v' = \frac{dv}{dx}, \tag{5.15}$$

$$v(0) = u_0, \quad v(l) = u_l. \tag{5.16}$$

Now take $\lambda = \lambda(x)$ to be a Lagrange multiplier corresponding to the continuous constraint (5.15) and μ to be a Lagrange multiplier corresponding to the constraint (5.16). When formulating a Lagrange function, an integral over the segment $(0, l)$ corresponds to the continuous constraint (5.15). Thus, we have the problem:

Problem 5.6. Compute the stationary point of the functional

$$\begin{aligned}\mathcal{L}(v, v', \lambda, \mu_0, \mu_l) =& J(v, v') + \int_0^l \left(\frac{dv}{dx} - v'\right) \lambda dx \\ & - \mu_0[v(0) - u_0] + \mu_l[v(l) - u_l]\end{aligned} \tag{5.17}$$

for all variables without any constraints.

The stationarity condition of the functional (5.17) is that all the functional derivatives are zero. First, we transform the functional (5.17) using integration by parts to obtain

$$\begin{aligned}\mathcal{L}(v, v', \lambda, \mu_0, \mu_l) =& \int_0^l \left(\frac{1}{2}ES(v')^2 - Fv - v\frac{d\lambda}{dx} - v'\lambda\right) dx \\ & - \mu_0[v(0) - u_0] + \mu_l[v(l) - u_l] + v(l)\lambda(l) - v(0)\lambda(0).\end{aligned} \tag{5.18}$$

Let a derivative be denoted with a prime, an argument of differentiation with a subscript, and a stationary point with a wave, and let a direction of differentiation be without marks. Then, the first stationarity condition is given by

$$\mathcal{L}'_v(\tilde{v}, v; \tilde{v}', \tilde{\lambda}, \tilde{\mu}_0, \tilde{\mu}_l) = 0 \quad \forall v. \tag{5.19}$$

Problem 5.7. We allow v to be arbitrary in the equation (5.19), so obtaining three conditions

$$\frac{d\lambda}{dx} + F = 0, \tag{5.20}$$

$$\tilde{\mu}_0 + \tilde{\lambda}(0) = 0, \quad \tilde{\mu}_l + \tilde{\lambda}(l) = 0. \tag{5.21}$$

Analogously, by allowing v', $\lambda(l)$ and $\lambda(0)$ to vary arbitrarily, we obtain

$$ESv' - \tilde{\lambda} = 0, \tag{5.22}$$

$$\frac{d\tilde{v}}{dx} - \tilde{v}' = 0, \tag{5.23}$$

$$\tilde{v}(0) - u_0 = 0, \quad \tilde{v}(l) - u_l = 0. \tag{5.24}$$

According to the Hilbert principle, we add the conditions, (5.23) and (5.24) as a constraint for Problem 5.6, thus again obtaining Problem 5.5. If we apply additional constraints (5.20)–(5.22) to the varying functions, then we obtain the so-called Friedrichs transformation (see the definition in [CH53]). This transformation results in the problem:

Problem 5.8. Find the maximum value of the functional:

$$J^*(\lambda) = -\frac{1}{2}\int_0^l \frac{1}{ES(x)}\lambda^2 dx - \lambda(0)u_0 + \lambda(l)u_l \longrightarrow \max_{\lambda} \tag{5.25}$$

with the additional condition

$$\frac{d\lambda}{dx} + F = 0. \tag{5.26}$$

The forced conditions (5.16) of Problem 5.5 become the natural ones in Problem 5.8.

We point out the form of an intermediate problem, which is useful for the better interpretation and summarizing of further reading. For this purpose, we denote the conditions (5.22) and (5.24) as additional conditions of Problem 5.6. We get the problem:

Problem 5.6a. Find the stationary value of the functional:

$$\mathcal{L}_a(v, v') = \int_0^l \left\{ \frac{1}{2} ES(v')^2 - Fv + ESv' \left(\frac{dv}{dx} - v' \right) \right\} dx. \tag{5.27}$$

Now, we denote a new functional argument

$$p = ESv' \tag{5.28}$$

and an auxiliary function

$$\Phi(x, v, p) = pv' - \frac{1}{2} ES(v')^2 - Fv = \frac{1}{2ES} p^2 - Fv. \tag{5.29}$$

Then Problem 5.6a becomes the so-called *canonical problem*:

Problem 5.6b. Compute the stationary value of the functional:

$$\mathcal{L}_\delta(v, p) = \int_0^l \left\{ p \frac{dv}{dx} - \Phi(x, v, p) \right\} dx \tag{5.30}$$

with the additional condition

$$v(0) = u_0, \quad v(l) = u_l. \tag{5.31}$$

The transformation (5.28)–(5.29) is called the *Legendre transformation*. Note that the functional $\mathcal{L}_\delta(v, p)$ has a minimum on v and maximum on p. If we apply minimization on v then

$$\min_v \mathcal{L}_\delta(v, p) = J^*(p). \tag{5.32}$$

The functional $J^*(p)$ is determined according to the formula (5.25). Also, we note that

$$\min_v J(v) = \min_v \max_p \mathcal{L}_\delta(v, p) = \max_p \min_v \mathcal{L}_\delta(v, p) = \max_p J^*(p). \tag{5.33}$$

Moreover, the additional constraint (5.26) is applied in the last problem.

Note that all this reasoning also applies in the case where we need to minimize an arbitrary functional of v and v', not the particular functional (5.14).

5.1.3 Dirichlet problem for the Poisson equation

We demonstrated in Section 2.3 that problems of the torsion of prismatic rods with the Saint-Venant hypothesis, and the problem of membrane bending, are modeled by BVPs for the Laplace and Poisson equations. To keep matters clear, consider the Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Lambda \subset \mathbb{R}^2, \tag{5.34}$$

$$u|_\Gamma = 0, \quad \Gamma = \partial\Lambda. \tag{5.35}$$

From Section 2.3 it follows that the problems, (5.34) and (5.35) is equivalent to minimization of the functional

$$J(v) = \frac{1}{2}\int_\Lambda |\nabla v|^2 \, d\Lambda - \int_\Lambda f v \, d\Lambda, \quad v \in V, \tag{5.36}$$

where $V = H_0^1(\Omega)$.

Meaningful results are obtained when the function v and its partial derivatives in the functional (5.36) are considered to be independent, as in the previous problem. To generalize, we replace the boundary condition (5.35) by the nonhomogeneous boundary condition. Thus, the problem can be stated as follows:

Problem 5.9. Compute the minimum of the functional

$$J(v,p) = \frac{1}{2}\int_\Lambda |p|^2 \, d\Lambda - \int_\Lambda f v \, d\Lambda \tag{5.37}$$

with the constraints

$$v|_\Gamma = g(x,y), \quad (x,y) \in \Gamma \equiv \partial\Lambda, \tag{5.38}$$

$$p - \nabla v = 0, \quad (x,y) \in \Lambda. \tag{5.39}$$

Applying the conditions (5.38) and (5.39) to functional (5.37) and using the method of Lagrange multipliers, we formulate the problem:

Problem 5.10. Compute the stationary value of the functional

$$\mathcal{L}(v,p,\rho,\lambda) = \int_\Lambda \left[\frac{1}{2}|p|^2 + \lambda \cdot (\nabla v - p) - f v\right] d\Lambda - \int_\Gamma \rho(v-g)\, d\Gamma \tag{5.40}$$

with the unconstrained variables v, p, ρ, λ.

Using the same methods as in the previous problem, we conclude that the following conditions are satisfied at the stationary point:

$$p - \lambda = 0, \tag{5.41}$$

$$\operatorname{div} \lambda = f, \tag{5.42}$$

$$(\lambda \cdot \nu - \rho)|_\Gamma = 0, \tag{5.43}$$

$$\nabla v - p = 0, \tag{5.44}$$

$$(v - g)|_\Gamma = 0. \tag{5.45}$$

Problem 5.11. Compute the conditions (5.41)–(5.45).

Applying the conditions (5.44) and (5.45) as additional conditions for Problem 5.10, we return to Problem 5.9. If we apply the conditions (5.41)–(5.43) as additional conditions, then the Friedrichs transformation is executed, which leads to the problem:

Problem 5.12. Find the maximum of the functional

$$J^*(p) = -\frac{1}{2}\int_{\Lambda} |p|^2 \, d\Lambda + \int_{\Gamma} p \cdot \nu g \, d\Gamma \tag{5.46}$$

on a set of vectors p, satisfying the following equation in the domain Λ:

$$\operatorname{div} p = f. \tag{5.47}$$

Note that the forced boundary condition (5.38) becomes the natural one. When solving Problem 5.12, it is satisfied automatically. Conversely, if v satisfies the Neumann condition on part of the boundary then this condition appears as a forced boundary condition for Problem 5.12.

5.2 Equilibrium, mixed and hybrid variational principles in the theory of elasticity

5.2.1 Castigliano principle and the Reissner principle

Consider the general mixed problem (2.228)–(2.230), which was shown to be equivalent to the problem of functional minimization (2.233) with the constraint (2.229). (The constraint $v \in V$ is not significant in the current formal transformations.) Let us consider the displacement field v and the strain field ε_{ij} to be independent arguments of the functional (2.233), and replace the constraint (2.229) with a homogeneous one. Thus, we can formulate the problem:

Problem 5.13. Calculate the minimum of the functional

$$J(\hat{\varepsilon}, v) = \frac{1}{2}\int_{\Omega} a_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \, d\Omega - \int_{\Omega} \rho F \cdot v \, d\Omega - \int_{\Sigma_\sigma} P \cdot v \, d\Sigma \tag{5.48}$$

with the additional constraints within the domain Ω

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{5.49}$$

and on its boundary

$$v|_{\Sigma_u} = g(x), \quad x \in \Sigma_u. \tag{5.50}$$

Include the constraints (5.49) and (5.50) in the functional (5.48) using a Lagrange multiplier. (The first multiplier, denoted as $\hat{\sigma}$, is a tensor field in the domain Ω and the second one, denoted as μ, is a vectorial field in Σ_u.) Thus, we have the problem:

Problem 5.14. Find the stationary value of the functional

$$\mathcal{L}(\hat{\varepsilon}, v, \hat{\sigma}, \mu) = \int_{\Omega} \left\{ \frac{1}{2} a_{ijkl} \varepsilon_{kl} \varepsilon_{ij} - \sigma_{pq} \left[\varepsilon_{pq} - \frac{1}{2}(v_{p,q} + v_{q,p}) \right] - \rho F \cdot v \right\} d\Omega - \int_{\Sigma_\sigma} P \cdot v \, d\Sigma - \int_{\Sigma_u} \mu \cdot (v - g) \, d\Sigma \tag{5.51}$$

with respect to the variables $(\hat{\varepsilon}, v, \hat{\sigma}, \mu)$ without any constraints.

Using the formula

$$\frac{1}{2} \int_{\Omega} \sigma_{ij}(v_{i,j} + v_{j,i}) \, d\Omega = - \int_{\Omega} \sigma_{ij,j} v_i \, d\Omega + \int_{\Sigma} \sigma_{ij} v_i \nu_j \, d\Sigma \tag{5.52}$$

and setting the functional derivatives to zero, we obtain the problem:

Problem 5.15.

$$-\sigma_{ij} + a_{ijkl}\varepsilon_{kl} = 0, \tag{5.53}$$

$$\sigma_{ij,j} + \rho F_i = 0, \tag{5.54}$$

$$(-P_i + \sigma_{ij}\nu_j)|_{\Sigma_\sigma} = 0, \tag{5.55}$$

$$(-\mu_i + \sigma_{ij}\nu_j)|_{\Sigma_u} = 0, \tag{5.56}$$

$$(u - g)|_{\Sigma_u} = 0, \tag{5.57}$$

$$\varepsilon_{ij} - (u_{i,j} + u_{j,i})/2 = 0. \tag{5.58}$$

It follows from the condition (5.53) that the Lagrange multiplier $\hat{\sigma}$ is the stress tensor (hence we denote it as $\hat{\sigma}$) and the multiplier μ is the surface traction at the boundary Σ_u of the body Ω.

Incorporating (5.57) and (5.58) as additional conditions for Problem 5.14, we return to Problem 5.13. If we use (5.53)–(5.56) as additional constraints, we obtain the problem:

Problem 5.16. Find the maximum of the functional

$$J^*(\hat{\sigma}) = -\frac{1}{2} \int_{\Omega} A_{ijkl}\sigma_{kl}\sigma_{ij} \, d\Omega + \int_{\Sigma_u} \sigma_{ij}\nu_j g_i \, d\Sigma \tag{5.59}$$

on a set of stress fields, satisfying the equilibrium condition

$$\sigma_{ij,j} + \rho F_i = 0, \quad x \in \Omega, \tag{5.60}$$

$$\sigma_{ij}\nu_j|_{\Sigma_\sigma} = P_i. \tag{5.61}$$

The tensor A_{ijkl}, which emerged during the process of eliminating variables ε_{ij}, is the compliance modulus tensor. Note also that the statement that the stationary point of the functional J^* is maximum, is proven as well, as in Problem 5.1. In the demonstration of this statement it can also be proved that the stationary point of the intermediate functional $\mathcal{L}$ is a saddle point –

it has a minimum on the "old" variables and a maximum on the "new" ones (the Lagrange multipliers).

The variational principle, given in Problem 5.16, is called the *Castigliano Variational Principle* (in some cases, typically in numerical analysis, it is called the *Equilibrium Variational Principle*) as mentioned in Section 2.3.

The intermediate (mixed) variational principle, given in Problem 5.14, is called the *Hu–Washizu principle* and arises as the result of the substitution of (5.52) into (5.51). This principle is not the only one possible, as it is not necessary to eliminate the variables from Problems 5.14 and 5.15 in the same order as was done before. In particular, Reissner proposed the stress tensor field $\hat{\sigma}$ and the displacement vector field v to be used as independent functional arguments. Applying conditions (5.53) and (5.58), we eliminate the other variables. This leads to the problem of computing the stationary point of the functional

$$\begin{aligned} R(\hat{\sigma}, v) = & \int_\Omega [\sigma_{ij}\varepsilon_{ij}(v) - \frac{1}{2}A_{ijkl}\sigma_{kl}\sigma_{ij} - \rho F \cdot v]\, d\Omega \\ & - \int_{\Sigma_\sigma} P \cdot v \, d\Sigma - \int_{\Sigma_u} \sigma_{ij}\nu_j(v_i - g_i)\, d\Sigma, \end{aligned} \tag{5.62}$$

which is called the *Reissner functional* (in some cases the *Hellinger–Reissner functional*). The advantage of this functional lies in the fact that no constraints are applied to the varying fields $\hat{\sigma}$ and v. The equilibrium condition inside the domain and on its boundary as well as the boundary condition on Σ_u are natural. The functional (5.62) is used in aviation design.

Note that the method of transforming the variational statements (principles) allows us to construct an exhaustive collection of variational principles corresponding to a problem under consideration. First, choose the set of variables (σ, $\hat{\varepsilon}$, $u|_\Omega, \mu|_{\Sigma_u}$, $u|_{\Sigma_\sigma}$) which describes the given problem. Then, choose a subset of these variables and eliminate the remaining variables to obtain the corresponding variational principle.

Note also that for the problems under consideration we can choose the basic independent arguments as a displacement and/or traction at the boundary. In such a case we must satisfy the equilibrium equation in the domain Ω. This problem is solved usually by the fundamental solution for the corresponding differential operator. Then the variational principle contains a singular integral. The functions to be varied in these principles belong to the trace spaces, see Section 1.4.

5.2.2 Hybrid principles

One of the fruitful directions in the development of the FEM is the "hybrid finite element method" [TP69, CL91, CL96]. In this method the domain Ω is considered as a union of its subdomains (finite elements) with an appropriate contact condition on the subdomain boundaries, e.g., the continuity condition for the surface displacements and/or tractions. We formulate a variational principle for such of subdomain and sum the integral over all the subdomains and their boundaries taking into account the boundary conditions.

Let us obtain a hybrid functional formation for the Dirichlet problem for the Laplace equation

$$-\Delta u(x) = 0, \quad x \in \Omega,\ u|_{\Sigma} = g. \tag{5.63}$$

Decompose the domain Ω onto the subdomains T_l, the boundaries of which are denoted by ∂T_l. Let us examine a subdomain T_l, and assume temporarily that the normal derivative of u on ∂T_l is known. The solution of the problem (5.63) in the domain T_l is equivalent to the minimization of the functional

$$J_l(v) = \frac{1}{2}\int_{T_l} |\nabla v|^2 \, d\Omega - \int_{\partial T_l} \frac{\partial v}{\partial \nu} v \, d\Sigma. \tag{5.64}$$

If $\partial T_l \subset \Sigma$ then the surface integral on this segment is equal to zero and it is necessary to satisfy the condition (5.39).

Assume that the function v can be discontinuous at the intersection of the boundaries of the subdomains T_l. (Recall that the normal derivatives $\partial v/\partial \nu$ for the adjacent elements are opposite in sign.) Sum the equality (5.64) on all the subdomains T_l. Including the boundary condition into the functional and using the method of Lagrange multipliers, we obtain the problem: find the stationary point of the functional

$$\begin{aligned}\tilde{J}\left(v, \left.\frac{\partial v}{\partial \nu}\right|_{\partial T_l}\right) &= \frac{1}{2}\sum_l \int_{T_l} |\nabla v|^2 \, d\Omega \\ &- \sum_{l, \partial T_l \notin \Sigma} \int_{T_l} \frac{\partial v}{\partial \nu} v \, d\Sigma - \sum_{l, \partial T_l \in \Sigma} \int_{\partial T_l} \frac{\partial v}{\partial \nu}(g - v) \, d\Sigma, \end{aligned} \tag{5.65}$$

where v is a piece-wise function defined separately for each of the subdomains T_l, and $\partial v/\partial \nu$ is the function defined on the boundaries ∂T_l of the subdomain T_l. Note that inside the body v is unknown, it appears twice in the system (5.65) and it disappears if it is continuous in the transition through the boundary ∂T_l.

The problem of finding the stationary point of the functional (5.65) is the *first hybrid variational principle*. When the function v is continuous in crossing the boundaries of the elements (subdomains) T_l, the integrals on the internal boundaries will be zero. If, in addition, we include the boundary condition on Σ as a constraint, then we return to the standard Lagrange variational principle. The advantage of the considered transformation is the possibility of not having to satisfy the continuity requirement of the function v on the internal boundaries – the continuity condition is natural for this variational principle. This statement is demonstrated by means of the differentiation of the functional (5.65) with respect to the variable $\partial v/\partial \nu$ and setting this derivative zero.

The *second hybrid variational principle* is obtained in the case where the functional (5.46) defined in the subdomain T_l is used as the starting point. Assuming that the function v is continuous across the boundaries of the elements and the derivative $\partial v/\partial \nu$ can be discontinuous and repeating these calculations and reasoning, we conclude that the problem (5.63) is equivalent to the problem of finding the stationary point of the functional

$$\tilde{J}^*(p, v|_{\partial T_l}) = -\frac{1}{2}\sum_l \int_{T_l} |p|^2 \, d\Omega + \sum_{l,\partial T_l \notin \Sigma} \int_{\partial T_l} p_i \nu_i v \, d\Sigma + \sum_{l,\partial T_l \in \Sigma} \int_{\partial T_l} p_i \nu_i g \, d\Sigma \tag{5.66}$$

on a set of functions satisfying the condition

$$\operatorname{div} p = 0 \tag{5.67}$$

in every subdomain T_l.

Now consider the problems of the linear theory of elasticity (2.228)–(2.230). Decompose the domain Ω into subdomains T_l and assume temporarily that the tractions t on the boundaries ∂T_l are known. Suppose temporarily that the displacements at the surface ∂T_l are known. Then the problem of determining the stresses and strains in the domain T_l is equivalent to the minimization problem for the functional with respect to the variable v:

$$J_l(v) = \frac{1}{2}\int_{T_l} a_{ijkh}\varepsilon_{kh}(v)\varepsilon_{ij}(v) \, d\Omega - \int_{T_l} \rho F \cdot v \, d\Omega - \int_{\partial T_l} t \cdot v \, d\Sigma, \tag{5.68}$$

where $t = t_i k_i = \sigma_{ij}\nu_j k_i$ is the vector of the surface tractions on the boundary of the element. Suppose that the vectors t are equal and opposite on the common parts of the boundary ∂T_l of two adjacent elements, and that the displacements v can be discontinuous. Sum the expressions (5.68) for all the elements T_l and we obtain the problem: find the stationary value of the functional

$$\tilde{J}(v, t|_{\partial T_l}) = \sum_l \int_{T_l} \left[\frac{1}{2} a_{ijkh}\varepsilon_{kh}\varepsilon_{ij}(v) - \rho F \cdot v\right] d\Omega - \sum_{\partial T_l \in \Sigma_u} \int_{\partial T_l} t \cdot g \, d\Sigma - \sum_{\partial T_l \in \Sigma_\sigma} \int_{\partial T_l} P \cdot v \, d\Sigma - \sum_{\partial T_l \in \Sigma_u} \int_{\partial T_l} t \cdot v \, d\Sigma \tag{5.69}$$

on the variables $v \in T_l$, $t \in \partial T_l \subset \Sigma_u$, $v \in \partial T_l \subset \Sigma_\sigma$, $(t, v) \in \partial T_l \subset \Omega$.

If the Castigliano functional (5.59) is taken as the starting point, then, with the continuity requirement for the displacement vector on the whole of the domain, we obtain, repeating the above reasoning, the following hybrid variational principle: The problems (2.228)–(2.230) is equivalent to finding the stationary point of the functional

$$\tilde{J}^*(\hat{\sigma}, v|_{\partial T_l}) = -\sum_l \int_{T_l} \frac{1}{2} A_{ijkh}\sigma_{kh}\sigma_{ij}\, d\Omega - \sum_{\partial T_l \notin \Sigma} \int_{\partial T_l} \sigma_{ij}\nu_j v_i\, d\Sigma$$
$$+ \sum_{\partial T_l \in \Sigma_\sigma} \int_{\partial T_l} (P_i - \sigma_{ij}\nu_j) v_i\, d\Sigma + \sum_{\partial T_l \in \Sigma_u} \int_{\partial T_l} \sigma_{ij}\nu_j g_i\, d\Sigma \tag{5.70}$$

on the set of functions satisfying the equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i = 0 \tag{5.71}$$

in each subdomain T_l.

The *third version of the hybrid variational principle* can be obtained if we use the Reissner functional as the starting point. We leave the derivation of this principle to the reader as an exercise.

Notice that the hybrid variational principles are very useful for computing the stress-deformed state of ideally plastic bodies when the unknown functions can be discontinuous [Rep89] on a plastic flow surface.

5.2.3 Kinematic constraints in the domain (Herrmann principle)

A typical example of a kinematic constraint in the domain is the incompressibility condition used in models of such materials as rubber, where the shear modulus is 3–4 orders less by magnitude than the bulk modulus, and thus the Poisson ratio differs only slightly from the value 1/2. Another example concerns materials, widely used nowadays, reinforced by fibers or membranes with a high modulus of elasticity of elongation. The lengthening in the direction of reinforcement is close to zero and should be taken to zero. Numerical analysis of materials using the actual value of the modulus of elasticity is difficult, because the coefficients in the system of equations can differ by thousands and tens of thousands one from another. This leads to numerical instability and accumulation of errors.

The approach to the solution of such problems is based on the principle of the simultaneous use of displacement and force-like variables. The best known and widely used approach is the *Herrmann method* and associated with it the Herrmann principle [Her65] useful for the analysis of incompressible and weakly compressible materials.

The Herrmann functional is obtained by using the method of the Lagrange multipliers. For this purpose, we first introduce the new unknown function H (as was done in [Her65]) according to the formula

$$-p = \frac{1}{3}\sigma_{kk} = \frac{2\mu(1+\nu)}{3} H, \tag{5.72}$$

where, as before, p is the average pressure, μ is the shear modulus, and ν is the Poisson ratio. Decompose the tensors $\hat{\sigma}$ and $\hat{\varepsilon}$ into spherical and deviator components (according to the formulae (3.111)). Using the Hooke law, we state that

$$\theta = (1 - 2\nu)H. \tag{5.73}$$

In the case of incompressibility $\nu = 1/2$ the formula (5.73) yields the well-known incompressibility condition

$$\operatorname{div} u \equiv \theta = 0. \tag{5.74}$$

The formulae (5.72) and (5.73) allow transformation of the Hooke law to

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + 2\mu\nu H\delta_{ij}, \tag{5.75}$$

where δ_{ij} is the Kronecker symbol.

Considering v and H to be independent functional arguments and using the Lagrange variational principle (the functional (2.233)), we conclude that the problem (2.228)–(2.230) is equivalent to the problem of searching for the stationary point of the functional

$$J(v, H) = \int_\Omega [\mu\varepsilon_{ij}\varepsilon_{ij} + \mu\nu H\theta - \rho F \cdot v]\, d\Omega - \int_{\Sigma_\sigma} P \cdot v\, d\Sigma \tag{5.76}$$

with the additional constraint (5.73). Include the constraints (5.73) into the functional (5.76) using the Lagrange multiplier κ. It can be demonstrated that, first, the constraint (5.73) appears to be natural if $\kappa = \mu\nu H$, and, secondly, the problem is equivalent to searching for the stationary point of the functional

$$J(v, H) = \int_\Omega \{\mu[\varepsilon_{ij}(v)\varepsilon_{ij}(v) + 2\nu H\theta - \nu(1-2\nu)H^2] - \rho F \cdot v\}\, d\Omega - \int_{\Sigma_\sigma} P \cdot v\, d\Sigma \tag{5.77}$$

without any additional constraints on the fields v and H.

The condition of the absence of elongations along a line or a surface can be taken into account by a similar method. The methods of computation (for finite deformations) are described in [Ogd84].

5.3 Young–Fenchel–Moreau duality transformation

5.3.1 Definition of duality transformation

This section is devoted to the "duality transformation" [ET74, Tik86] which permits the generalization of the methods of the previous paragraph to the contact problems.

Let us first examine the geometric interpretation of a duality transformation. For this purpose, consider the problem: find the minimum value of the function

$$y = f(x) \tag{5.78}$$

everywhere on the straight line $-\infty \leq x \leq +\infty$.

Recall the definition of the convexity (see Section 3.2) and the definition of a convex set.

Definition 5.17. *The function $f(x)$ is called convex if for two arbitrary values x_1, x_2 and any number $t \in [0,1]$ the inequality*

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \tag{5.79}$$

holds. If the inequality is strict for all $x_1 \neq x_2$ and for all t such that $0 < t < 1$, the function $f(x)$ is called strictly convex.

Definition 5.18. *A set $K \subset V$ of a functional space V is called convex if for two arbitrary elements $v_1, v_2 \in K$ and any number $t \in [0,1]$ we have the inclusion*

$$v = (1-t)v_1 + tv_2 \in K.$$

Definition 5.19. *A functional $J(v)$ is a proper functional if it is not equal to $+\infty$.*

We consider minimization problems for a convex function or convex functional on the convex set K assuming the minimized function or functional equal $+\infty$ outside the set K. The last hypothesis does not violate the convexity property, but it violates the differentiability. If we extend the function, differentiable inside the subset K, to a function equal to $f(x)$ inside K and $+\infty$ outside K, we obtain a function which is nondifferentiable on the boundaries of the set K. This is the main reason for the generalization of the concept of derivative to a set of nonsmooth functions.

To explain the essential idea, consider first a 1D problem. Let the graph of a convex function be represented by the curve $f(x)$ in Figure 5.1. Let us consider the tangent to the curve $f(x)$ at the point $(\tilde{x}, f(\tilde{x}))$, with the equation

$$y = mx + c \tag{5.80}$$

and a straight line which is located below this curve, i.e.,

$$f(x) \geq mx + c \quad \forall x \tag{5.81}$$

and parallel to the tangent (see Figure 5.1).

The length of the segment PM is equal to $PM = f(x) - mx - c$. On the tangent we have the equation:

$$f(\tilde{x}) = m\tilde{x} + \tilde{c}.$$

Then $\tilde{c} = f(\tilde{x}) - m\tilde{x}$ and

$$\tilde{c} = \inf_x [f(x) - mx] = -\sup_x [mx - f(x)] = -f^*(m). \tag{5.82}$$

The transformation given by (5.82) for the different values m is denoted by $f^*(m)$ and called the *duality transformation* of the function $f(x)$. So, by definition

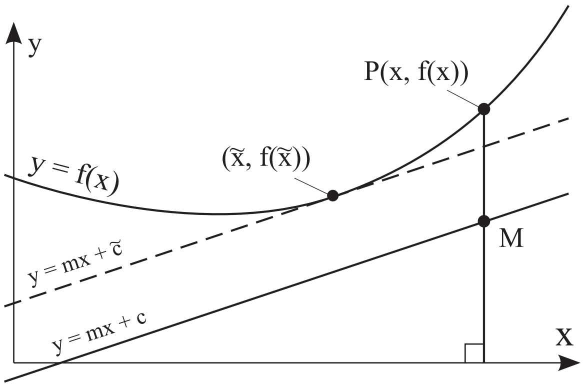

Fig. 5.1. Illustration of the duality transformation

$$f^*(m) = \sup_x [mx - f(x)]. \tag{5.83}$$

If the function $f(x)$ is differentiable at the point x_0, then

$$m = f'(x_0). \tag{5.84}$$

Comparing the formulae (5.83) and (5.84) with the formulae (5.28) and (5.29), we see that the transformation (5.83) for the differentiable function $f(x)$ coincides with the Legendre transformation [CH53].

If the function $f(x)$ is nondifferentiable at the point x_0, then the calculation by the formula (5.83) gives a set $\{m\}$ of coefficients m, which is called the *subdifferential* of the function $f(x)$ at the point x_0 and denoted by $\partial f(x_0)$. For example, if $f(x) = |x|$ then $\partial f(0) \in [-1, +1]$, see Example 5.23 in Section 5.3.2.

We give now the generalization of this definition to the functional. Consider, for generality, a Banach space V and its dual V^*, i.e., the space of linear functionals on V. Recall that in Chapter 1 we introduced the notation v^* for a linear functional on V and the notation $\langle v, v^* \rangle$ for the value of the functional V^* on the element $v \in V$.

Let $J(v)$ be a functional on the space V and v^* an element of the dual space V^*. Define the functional $J^*(v^*)$ by

$$J^*(v^*) = \sup_{v \in V} [\langle v^*, v \rangle - J(v)]. \tag{5.85}$$

The functional $J^*(v^*)$ is called the *Young–Fenchel–Moreau transformation* of the functional $J(v)$. Sometimes $J^*(v^*)$ is called the *polar* of the functional $J(v)$. From the definition (5.85) it follows the properties:

1. It holds
$$J^*(0) = -\inf J(u). \tag{5.86}$$
2. If $J \le F$, then $J^* \ge F^*$.

3. For any set of functionals $\{J_i\}_{i=1}^N$ on V the following conditions are satisfied:
$$(\inf_i J_i)^* = \sup_i J_i, \quad (\sup_i J_i)^* \leq \inf_i J_i^*. \tag{5.87}$$
4. For any number $\lambda > 0$ the following equality holds:
$$(\lambda J)^*(u^*) = \lambda J^*(u^*/\lambda). \tag{5.88}$$
5. For the arbitrary number α the following equality holds:
$$(J + \alpha)^* = J^* - \alpha. \tag{5.89}$$
6. For any element $v \in V$ the following equality holds:
$$(J(u - v))^* = J^*(u^*) + \langle u^*, v\rangle. \tag{5.90}$$
7. If $J(v)$ is the point-wise upper boundary of a family of affine functionals (i.e., functionals of the type $L(v) + a$, where $L(v)$ is the continuous linear functional, a is a number), then
$$J^{**}(v) = \sup_{v^* \in V} [\langle v^*, v\rangle - J^*(v^*)] = J(v). \tag{5.91}$$

We now give some examples of the Young–Fenchel–Moreau transformation.

Example 5.20. Let v_0^* be a fixed element of the space V^* and α a number. Choose
$$J(v) = \langle v_0^*, v\rangle + \alpha. \tag{5.92}$$
Then
$$J^*(v^*) = \sup_{v \in V} [\langle v^* - v_0^*, v\rangle - \alpha] = \begin{cases} -\alpha, & v^* = v_0^*, \\ +\infty, & v^* \neq v_0^*. \end{cases} \tag{5.93}$$

Example 5.21. Let K be a subset of the space V. Choose
$$J(v) = \delta(v|K) = \begin{cases} 0, & v \in K, \\ +\infty, & v \notin K. \end{cases} \tag{5.94}$$
The function $\delta(v|K)$ is called the *indicatrix-function* or briefly *indicatrix* of the set K. We obtain
$$J^*(v^*) = \sup_{v \in K} \langle v^*, v\rangle \equiv s(v^*|K). \tag{5.95}$$
The function s is called the *support function* of the set K.

Example 5.22. Let the number $\alpha \in (0, +\infty)$ and
$$J(v) = \frac{1}{\alpha}\|v\|^\alpha. \tag{5.96}$$
Then
$$J^*(v^*) = \frac{1}{\alpha^*}\|v^*\|_{V^*}^{\alpha^*}, \tag{5.97}$$
where the number α^* satisfies the equation
$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1. \tag{5.98}$$

5.3.2 General definition of subdifferential and subgradient

The concepts of a subdifferential and a subgradient emerged in the process of generalizing the derivative using the comparison of the formulae (5.83) and (5.84). They are useful as instruments for finding the necessary and sufficient conditions of the extremum of nondifferentiable functionals and for the formulation and justification of approximate solution methods.

Let the functional $F : V \to \mathbb{R}$ be convex, but not necessarily differentiable. The element $u^* \in V^*$ is called the *subgradient* of the functional F at the point u if

$$F(v) - F(u) \geq \langle u^*, v - u \rangle \tag{5.99}$$

holds for any elements $v \in V$. Recall that the angle brackets $\langle \cdot, \cdot \rangle$ denote a linear functional on V, more precisely, the bilinear form, which brings the spaces V and V^* into duality. The set of all the subgradients at the point u is called the *subdifferential* of the functional $F(u)$ at the point u and is denoted by $\partial F(u)$:

$$\partial F(u) = \{u^* \mid F(v) - F(u) \geq \langle u^*, v - u \rangle \ \forall v \in V\}. \tag{5.100}$$

The definition (5.99) shows that the linear (precisely, affine) functional

$$l(v) = F(u) + \langle u^*, v - u \rangle$$

bounds the given functional $F(v)$ from below and is tangential to the "graph" of the functional $F(v)$ at the point u. The meaning of the subgradient can be more clearly illustrated with the following example for a function of one variable.

Example 5.23. Let $V = \mathbb{R}$, $F(v) = |v|$. From the graph of the function $|v|$ one can see that this function is convex. The inequality (5.99) gives

$$|v| - |u| \geq u^*(v - u) \quad \forall v \in \mathbb{R}. \tag{5.101}$$

In $\mathbb{R}$ the angle bracket is simply a product. To find u^*, we must consider all the possibilities for the values u:

1. Let $u > 0$. Then $|u| = u$. If $v > 0$ then $|v| = v$ and we obtain

$$v - u \geq u^*(v - u). \tag{5.102}$$

The inequality (5.102) can be satisfied only if $u^* \leq 1$, $v - u > 0$. This inequality must be compatible with the case $v < 0$, when $|v| = -v$. Thus, from (5.102) we obtain

$$-v - u \geq u^*(v - u)$$

or

$$-v(1 + u^*) \geq u(1 - u^*) \quad \forall v < 0. \tag{5.103}$$

Choose $-v = u$ (it is acceptable because $u > 0$) and divide by u

$$1 + u^* \geq 1 - u^*.$$

It follows from this that $u^* \geq 1$, and we finally obtain

$$u^* = 1, \tag{5.104}$$

i.e., u^* is equal to the derivative of the linear function $F(u) = |u| = u$ for $u > 0$.

2. Let $u < 0$. Analogous reasoning demonstrates that in this case u^* is also equal to the derivative of the function $F(u) = |u| = -u$ if $u < 0$, i.e., $u^* = -1$.
3. Let $u = 0$. Then $|v| > u^* v$ for all v, and it follows from this that

$$-1 \leq u^* \leq +1. \tag{5.105}$$

Thus, when we move from the left to the right, i.e., the variable v increases, then in the passage through the point of nondifferentiability $u = 0$ the subgradient changes from the value of the derivative on the left to the derivative on the right. Therefore, we can see that the subdifferential take all the values between the derivative on the left and the derivative on the right.

Recall that we investigated the minimization problem for a convex functional $F(v)$ on the convex set $K \subset V$, and we extended $F(v)$ to all the space V by setting $F(v) = +\infty$ outside K. At the minimum point u we have the inequality:

$$F(v) \geq F(u), \quad \forall v \in V. \tag{5.106}$$

Comparing this inequality with the definition (5.99), we conclude that

$$0 \in \partial F(u) \tag{5.107}$$

and this formulation is the general formulation of a minimization problem without the differentiability requirement.

We now formulate some properties of the subdifferential procedure:

1. The subdifferential of the norm of the Banach space at the zero point coincides with the unit ball of the dual V^*. If $v \neq 0$, then

$$\partial \|v\| = \{v^* \in V^* \mid \|v^*\| = 1, \ \langle v^*, v \rangle = \|v\|\}. \tag{5.108}$$

This statement follows from the definition (5.100) of the subdifferential, with the formula

$$\|u^*\| = \sup_{\|v\|_V = 1} \langle u^*, v \rangle,$$

and from Definition 1.47 of the norm of a linear operator in the Banach space V.

2. Let $\delta(v|K)$ be the indicatrix of a convex set K. Then

$$\partial\delta(v|K) = \{v^* \in V^* \mid \langle v^*, w - v\rangle \le 0 \ \forall w \in K\}. \tag{5.109}$$

The set (5.109) for the fixed v is a cone, called the *cone of support functionals.*

3. For any pair J_1, J_2 of convex functionals on V, the following inclusion holds:

$$\partial(J_1 + J_2)(v) \supset \partial J_1(v) + \partial J_2(v). \tag{5.110}$$

If one of these functionals is continuous at a point of some set, in which another functional is finite, then

$$\partial(J_1 + J_2)(v) = \partial J_1(v) + \partial J_2(v), \tag{5.111}$$

which is called the *Moreau–Rockafellar theorem.*

4. The element $u^* \in \partial J(u)$ if and only if

$$J(u) + J^*(u^*) = \langle u^*, u\rangle. \tag{5.112}$$

5. If the functional J is Gâteaux-differentiable (see the definition (3.14)) at the point $u \in V$, then its subdifferential is $\partial J(u) = J'(u)$. If the functional J at the point $u \in V$ is continuous, finite, and has only one subgradient, then it is Gâteaux-differentiable at the point u and $\partial J(u) = F'(u)$.

The proofs can be found, e.g., in [ET74, Tik86].

5.3.3 Method of solving minimization (maximization) problems

The rest of this chapter is devoted to the construction of effective methods for the solution of the contact problems using the tools we have introduced.

Note first that the application of the duality transformation to an initial formulation $(\mathcal{P})$ of the contact problem as a minimization problem of $J(v)$ gives only the trivial relation (5.86), which does not give any new solution method. In order to obtain meaningful results in practice, we introduce the so-called *disturbance* of the functional $J(v)$.

Let $\Phi : V \times Y \to \mathbb{R}$ be a functional on the direct product of the space V and a new space Y. The elements of the space Y will be denoted by p or q. The more precise definition of the disturbance is as follows: the functional $\Phi(v, p)$,

$$\Phi : V \times Y \longrightarrow \mathbb{R},$$

is called the *disturbance* of the functional $J(v)$, if it is convex on the direct product of $V \times Y$:

$$\Phi(v, 0) = J(v). \tag{5.113}$$

The problem

$$\Phi(v, p) \longrightarrow \inf_{v \in V}, \tag{$\mathcal{P}_p$}$$

the solution to which depends on the parameter p, is called the *disturbance of the problem* $(\mathcal{P})$.

Let Y^* be dual to Y with respect to some bilinear form $\langle \cdot, \cdot \rangle_Y$. Then the spaces $V \times Y$ and $V^* \times Y^*$ are brought into duality using the bilinear form

$$\langle (v^*, p^*), (v, p) \rangle = \langle v^*, v \rangle_V + \langle p^*, p \rangle_Y. \tag{5.114}$$

Applying the duality transformation in the form (5.114), we get the functional

$$\Phi^*(v^*, p^*) = \sup_{v \in V} \sup_{p \in Y} [\langle v^*, v \rangle_V + \langle p^*, p \rangle_Y - \Phi(v, p)]. \tag{5.115}$$

By definition the problem

$$-\Phi^*(0, p^*) \longrightarrow \sup_{p^* \in Y} \tag{$\mathcal{P}^*$}$$

is called the *dual* or *adjoint* to the problem $(\mathcal{P})$.

For the functionals $\Phi(v, p)$, $\Phi^*(v^*, p^*)$, $h(p) = \inf_v \Phi(v, p)$, several interesting and important statements can be proved [ET74]. Since our main goal is to examine the algorithmic side of the problem, we describe the final theorem only. (The proof can be found in [ET74].)

Theorem 5.24. *Let V be the reflexive Banach space and let $\Phi \in \Gamma_0(V \times Y)$, where Γ_0 is the space of proper functionals, be the point-wise upper boundary of some family of continuous affine functionals.*

There exists an element $u_0 \in V$ for which the functional $p \to \Phi(u_0, p)$ is finite and continuous at the zero point (of the space Y). The following condition is satisfied (compare with the condition (3.252)):

$$\lim_{\|u\| \to +\infty} \Phi(u, 0) = +\infty. \tag{5.116}$$

Then

- *(i) The problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ have at least one solution (recall that we denote by $(\mathcal{P})$ the initial minimization problem)*
- *(ii) $\mathrm{Inf}_v J(v) = \sup_{p^*}[-\Phi^*(0, p^*)]$*
- *(iii) $\Phi(\tilde{u}, 0) + \Phi^*(0, \tilde{p}^*) = 0$, where $\tilde{u}$ is any solution of the problem $(\mathcal{P})$ and $\tilde{p}^*$ is any solution of the problem $(\mathcal{P}^*)$*
- *(iv) $(0, \tilde{p}^*) \in \partial\Phi(\tilde{u}, 0)$*

The relations (iii) and (iv) are called the *extremality relations.*

Thus, the dual problem $(\mathcal{P}^*)$ can be solved instead of the initial problem $(\mathcal{P})$. The solution of the initial problem is obtained through the solution of the dual problem using the extremality relations.

5.3.4 Transition to the saddle-point problem

The use of the Lagrangians is one way more to solve extremal problems.

Definition 5.25. *The functional $L : V \times Y^* \to \mathbb{R}$, given with the formula*

$$-L(v, p^*) = \sup_{p \in Y} [\langle p^*, p \rangle - \Phi(v, p)], \quad v \in V, \; p^* \in Y^*, \tag{5.117}$$

is called the Lagrangian of the problem $(\mathcal{P})$ with respect to the given disturbance $\Phi(v, p)$.

Using the stated constraints for the functional $J(v)$ and $\Phi(v, p)$, it is possible to prove [ET74] that

1. The functional $p^* \to L(v, p^*)$ is concave. Recall that concavity of the functional J means that the functional $-J$ is convex.
2. The functional $v \to L(v, p^*)$ is convex.

These new statements of extremal problems are based on the following series of equalities (derived from the definitions):

$$\begin{aligned}\Phi^*(v^*, p^*) &= \sup_{v \in V} \sup_{p \in Y} [\langle v^*, v \rangle + \langle p^*, p \rangle - \Phi(v, p)] \\ &= \sup_{v \in V} \{\langle v^*, v \rangle + \sup_{p \in Y} [\langle p^*, p \rangle - \Phi(v, p)]\} \\ &= \sup_{v \in V} [\langle v^*, v \rangle - L(v, p^*)]. \end{aligned} \tag{5.118}$$

It follows from (5.118) that

$$-\Phi^*(0, p^*) = \inf_{v \in V} L(v, p^*), \tag{5.119}$$

and, consequently, the problem $(\mathcal{P}^*)$ can be presented in the following form:

$$L(v, p^*) \longrightarrow \sup_{p^* \in Y^*} \inf_{v \in V} . \tag{5.120}$$

If we assume now that $\Phi \in \Gamma_0(V \times Y)$, then the duality transformation, applied to $\Phi^*(v^*, p^*)$, results in the "old" functional $\Phi(v, p)$. We obtain

$$\Phi(v, p) = \sup_{p^* \in Y^*} [\langle p^*, p \rangle + L(v, p^*)]. \tag{5.121}$$

It follows from here that

$$\Phi(v, 0) = \sup_{p^* \in Y^*} L(v, p^*). \tag{5.122}$$

Therefore, the problem $(\mathcal{P})$ can be written in the form

$$L(v, p^*) \to \inf_{v \in V} \sup_{p^* \in Y^*} . \tag{5.123}$$

Definition 5.26. *The point* $(\tilde{u}, \tilde{p}^*) \in V \times Y^*$ *is called a saddle point of Lagrangian* L*, if*

$$L(\tilde{u}, p^*) \le L(\tilde{u}, \tilde{p}^*) \le L(u, \tilde{p}^*) \quad \forall u \in V,\ \forall p^* \in Y^*. \tag{5.124}$$

The connection between the stated problems and the problems of finding a Lagrangian saddle point is given by Theorem 5.27.

Theorem 5.27. *If the conditions of Theorem 5.24 are satisfied, then the two following statements are equivalent:*

(i) $\tilde{u}$, $\tilde{p}^*$ *are solutions of the problems* $(\mathcal{P})$ *and* $(\mathcal{P}^*)$*, respectively*
(ii) The pair $\tilde{u}$, $\tilde{p}^*$ *is a saddle point of the Lagrangian* L

Thus, on the basis of these results it is possible to obtain for every disturbance $\Phi(v, p)$ two new formulations of the problem $(\mathcal{P})$ – the dual problem $(\mathcal{P}^*)$ and the problem of finding the Lagrangian L saddle point.

5.3.5 Special cases

Example 5.28. Compute the adjoint functional and the Lagrangian for the functional

$$J(v) = f(v, \Lambda v), \tag{5.125}$$

where $f : V \times Y \to \mathbb{R}$ and Λ is a linear operator from the space V into Y. Many of the functionals studied earlier belong to this set, as we show later.

The simplest disturbance of the functional (5.125) is the shift on the second argument:

$$\Phi(v, p) = f(v, \Lambda v - p), \quad p \in Y. \tag{5.126}$$

As shown above, the dual problem $(\mathcal{P})^*$ is associated with the following functional:

$$\Phi^*(0, p^*) = \sup_{v \in V} \sup_{p \in Y} [\langle p^*, p \rangle - f(v, \Lambda v - p)]. \tag{5.127}$$

To compute this, we assume that $q = \Lambda v - p$. We get

$$\begin{aligned} \Phi^*(0, p^*) &= \sup_{v \in V} \sup_{q \in Y} [\langle p^*, \Lambda v \rangle - \langle p^*, q \rangle - f(v, q)] \\ &= \sup_{v \in V} \sup_{q \in Y} [\langle \Lambda^* p^*, v \rangle + \langle -p^*, q \rangle - f(v, q)] \\ &= f^*(\Lambda^* p^*, -p^*). \end{aligned} \tag{5.128}$$

Hence, the problem, dual to the functional minimization problem (5.125), has the following form:

$$-f^*(\Lambda^* p^*, -p^*) \longrightarrow \sup_{p^* \in Y^*}. \tag{5.129}$$

We now consider the concrete situation, emerging in the Dirichlet problem for the Poisson equation,

$$f(v, \Lambda v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, d\Omega - \int_\Omega f(x)v(x)\, d\Omega \equiv G(\Lambda v) + F(v), \quad v \in H_0^1(\Omega). \tag{5.130}$$

We obtain

$$G(\Lambda v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, d\Omega, \tag{5.131}$$

$$\Lambda v = \nabla v : H^1(\Omega) \longrightarrow L^2(\Omega) \equiv Y, \tag{5.132}$$

$$F(v) = -\int_\Omega f(x)v(x)\, d\Omega. \tag{5.133}$$

When $\langle \cdot, \cdot \rangle$ is the scalar product in the space $L^2(\Omega)$, we have $Y^* = Y$.

We directly test that the equality

$$f^*(v^*, p^*) = F^*(v^*) + G^*(p^*) \tag{5.134}$$

occurs for the functional of the form (5.130) and that

$$F^*(v^*) = \begin{cases} 0, & u^* + f = 0, \\ +\infty, & u^* + f \neq 0, \end{cases} \tag{5.135}$$

$$G^*(p^*) \equiv G^*(y^*) = \frac{1}{2}\int_\Omega |y^*|^2\, d\Omega. \tag{5.136}$$

To calculate the functional (5.129), we first find the adjoint operator Λ^*:

$$\begin{aligned} \langle \Lambda v, y\rangle_Y &= \int_\Omega \nabla v \cdot y\, d\Omega = \int_{\partial\Omega} vy \cdot \nu\, d\Sigma - \int_\Omega v \operatorname{div} y\, d\Omega \\ &= -\int_\Omega v \operatorname{div} y\, d\Omega = \langle v, \Lambda^* y\rangle. \end{aligned} \tag{5.137}$$

The Green formula and the boundary condition $v|_\Sigma = 0$ are used in the series of the equalities (5.137). It follows from the formula (5.137) that

$$\Lambda^* y = -\operatorname{div} y. \tag{5.138}$$

We conclude from the formula (5.129) for the adjoint problem that the functional of the problem, which is dual to the problem of functional minimization (5.130), takes the form

$$-f^*(\Lambda^* p^*, -p^*) = -f^*(\Lambda^* y, -y) = -F^*(\Lambda^* y) - G^*(-y). \tag{5.139}$$

If we map out those elements for which $F^*(\Lambda^* y) = +\infty$, then the final result looks as follows:

$$-\frac{1}{2}\int_\Omega |y^*|^2\, d\Omega \longrightarrow \sup_{y;\ \operatorname{div} y = f}, \tag{5.140}$$

and coincides with the result obtained in Section 5.1.3 (if it is assumed there that $g = 0$). Thus, the Young–Fenchel–Moreau transformation for the functional (5.130) with the disturbance in the form (5.126) coincides with the Friedrichs transformation.

Example 5.29. Let the initial problem $(\mathcal{P})$ take the form

$$J(v) \to \inf_{v\in K\subset V;\, Bv\le 0}, \tag{5.141}$$

where K is a closed convex set and B is a map (probably, nonlinear) of K in Y having the convexity property. Besides for the existence of the solution, it is enough to assume, in addition, that

1. The reflection $u \to \langle p^*, Bu\rangle$ is lower semicontinuous for any $p^* \in Y^*$, $p^* \ge 0$,
2. The set of elements $v \in K$, for which $Bv \le 0$, is nonempty.

We use the following disturbance:

$$\Phi(v,p) = \begin{cases} J(v), & \text{if } v \in K,\ Bv \le p. \\ +\infty & \text{otherwise.} \end{cases} \tag{5.142}$$

For computation, it is proper to rewrite the definition (5.142) using the concept of the indicatrix function:

$$\Phi(v,p) = \tilde{J}(v) + \chi_{\varepsilon_p}(v), \tag{5.143}$$

$$\tilde{J}(v) = \begin{cases} J(v), & \text{if } v \in K, \\ +\infty, & \text{if } v \notin K, \end{cases} \tag{5.144}$$

where $\chi_{\varepsilon_p}(v)$ is the indicated function of the set

$$\varepsilon_p = \{v \mid v \in K,\ Bv \le p\}. \tag{5.145}$$

The definition tells us that

$$\Phi^*(0,p^*) = \sup_{v\in V}\sup_{p\in Y}[\langle p^*, p\rangle - \Phi(v,p)] = \sup_{\substack{v\in K,\ p\in Y \\ Bv\le p}}[\langle p^*, p\rangle - J(v)]. \tag{5.146}$$

We assume that $p = Bv + q$ and denote Ξ as the closed convex cone of the nonnegative elements in the space Y (see, e.g., [LVG02]). Then

$$\begin{aligned}\Phi^*(0,p^*) &= \sup_{v\in K}\sup_{q\in\Xi}[\langle p^*, Bv\rangle + \langle p^*, q\rangle - J(v)] \\ &= \chi_{\Xi^*}(-p^*) + \sup_{v\in K}[\langle p^*, Bv\rangle - J(v)]. \end{aligned} \tag{5.147}$$

Thus, the dual problem takes the form

$$\sup_{p^*\leq 0}\inf_{v\in K}\{-\langle p^*, Bv\rangle + J(v)\}. \tag{5.148}$$

We use the constraint $p^* \leq 0$.

This problem can be used for the problem of computing the saddle point of the Lagrangian (5.117). We calculate

$$\begin{aligned}-L(v,p^*) &= \sup_{p\in Y}[\langle p^*, p\rangle - \Phi(v,p)] = -\tilde{J}(v) + \sup_{p\in Y,\, p\geq Bv}\langle p^*, p\rangle \\ &= -\tilde{J}(v) + \langle p^*, Bv\rangle + \sup_{q\in Y,\, q\geq 0}\langle p^*, q\rangle. \end{aligned} \tag{5.149}$$

Taking into account the property $\sup\langle p^*, q\rangle$, which was already used in (5.147), we conclude that the pair $(\tilde{u}, \tilde{p}^*)$ is the solution of the initial and dual problems and satisfies the condition

$$J(\tilde{u}) - \langle p^*, B\tilde{u}\rangle \leq J(\tilde{u}) - \langle p^*, B\tilde{u}\rangle \leq J(u) - \langle \tilde{p}^*, Bu\rangle \quad \forall u\in K^*,\ \forall p^*\leq 0. \tag{5.150}$$

This is the problem of computing the saddle point of the Lagrangian.

Now we apply these results to contact problems. For short, the disturbance of the form (5.126) is called a *disturbance of the Castigliano form* and the disturbance of the form (5.142) a *disturbance of the Arrow–Hurwitz form.*

5.4 Applications of duality transformations in contact problems

5.4.1 Disturbance of the Arrow–Hurwitz form in the problem $(\mathcal{P})$

We consider the problem stated in the following form (see Theorem 4.2):

$$\begin{aligned}J(v) = \frac{1}{2}a(v,v) - L(v) &= \int_\Omega \left[\frac{1}{2}a_{ijkl}\varepsilon_{kl}(v)\varepsilon_{ij}(v) - \rho F\cdot v\right] d\Omega \\ &- \int_{\Sigma_\sigma} P\cdot v\, d\Sigma \longrightarrow \inf_{v\in K}, \end{aligned} \tag{5.151}$$

$$K = \{v \mid v\in V;\ v_N \leq \delta_N,\ \forall x\in\Sigma_c\} \tag{5.152}$$

We introduce the disturbance

$$\Phi = \Phi_1(v,p) = J(v) + \chi_\varepsilon(\{v,p\}), \tag{5.153}$$

where χ_ε is the indicatrix function of the set

$$\varepsilon = \{(v,p) \in V \times Y \mid v_N - \delta_N \leq p\}, \tag{5.154}$$

$$p \in Y = \{p \mid p = p(x),\ x \in \Sigma_c;\ p \in L^2(\Sigma_c)\}. \tag{5.155}$$

The given problem corresponds to the scheme of Example 5.29 if we consider the operator B to be the operator of computing the value of $v_N - \delta_N \in L^2(\Sigma_c)$ on the element $v \in V$. Using the result (5.148) of Example 5.29, we conclude that the dual problem corresponding to the disturbance (5.153) takes the form

$$J(v) + \int_{\Sigma_c} p^*(\delta_N - v_N)\, d\Sigma \longrightarrow \sup_{p^* \leq 0} \inf_{v \in V}. \tag{$\mathcal{P}_1^*$}$$

It is easy to show that the adjoint variable p^* coincides with the contact pressure σ_N, distributed on Σ_c.

5.4.2 Disturbance of the Castigliano form

We assume that

$$\Phi_2(v,\hat{p}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega a_{ijkl}[\varepsilon_{kl}(v) - p_{kl}][\varepsilon_{ij}(v) - p_{ij}]\, d\Omega - L(v) & \forall v \in K, \\ +\infty & \forall v \notin K. \end{cases} \tag{5.156}$$

Thus, $\hat{p}$ is the second-order tensor

$$Y = \{\hat{p} \mid p_{ij} = p_{ji};\ p_{ij} = p_{ij}(x),\ x \in \Omega\}. \tag{5.157}$$

We determine the space Y^* as the adjoint to Y with respect to the bilinear form

$$\langle \hat{p}, \hat{p}^* \rangle = \int_\Omega \hat{p} \cdot\cdot \hat{p}^*\, d\Omega = \int_\Omega p_{ij} p_{ij}^*\, d\Omega. \tag{5.158}$$

If we determine the operator $\Lambda : H^1(\Omega) \to L^2(\Omega)$ with the formula

$$\Lambda v = \frac{1}{2}(\nabla v + \nabla v^T) \tag{5.159}$$

then the problem (5.156) corresponds to the scheme of Example 5.28, and it follows from the formulae (5.127), (5.156), and (5.158) that $\hat{p}^* = \hat{\sigma}$ is the stress tensor. Using the formula (5.128) for the computations, we state that the dual problem takes the form

$$-\frac{1}{2}\int_\Omega A_{ijkl}\sigma_{kl}\sigma_{ij}\, d\Omega + \int_{\Sigma_c} \sigma_N \delta_N\, d\Sigma \longrightarrow \sup_{\hat{\sigma} \in M}, \tag{$\mathcal{P}_2^*$}$$

where

$$M = \{\hat{\sigma} \mid \operatorname{div}\hat{\sigma} + \rho F = 0;\ \hat{\sigma} \cdot \nu|_{\Sigma_\sigma} = P;\ \sigma_N|_{\Sigma_c} \leq 0;\ \sigma_T|_{\Sigma_c} = 0\}. \tag{5.160}$$

5.4.3 Combined disturbance (Lagrangians)

Let us consider the disturbance

$$\Phi = \Phi_3(v, p) = \Phi_2(v, \hat{p}_1) + \chi_\varepsilon(\{v, p_2\}), \tag{5.161}$$

where the functional Φ_2 is given by the formula (5.156) everywhere on V, not only on the set K. The constraint $v \in K$ is taken into account using the indicatrix function χ_ε. We assume that

$$p = \{\hat{p}, p_2\} \in Y_1 \otimes Y_2 \equiv Y, \quad \langle p^*, p\rangle = \langle \hat{p}_1^*, \hat{p}_1\rangle + \langle p_2^*, p_2\rangle. \tag{5.162}$$

Using the formula (5.115) for the computations and substituting the notation p_1^* with $\hat{\sigma}$, p_2^* on $\sigma_N = \sigma_{ij}\nu_i\nu_j$, we obtain the following dual statement:

$$\frac{1}{2}\int_\Omega A_{ijkl}\sigma_{kl}\sigma_{ij}\, d\Omega + \int_{\Sigma_c} \sigma_N(\delta_N - v_N)\, d\Sigma \longrightarrow \sup_{\hat{\sigma}\in M}\inf_{v\in V}. \tag{$\mathcal{P}_3^*$}$$

The problems $(\mathcal{P}_2^*)$ and $(\mathcal{P}_3^*)$ are difficult, regarding their approximate solution (e.g., by the FEM or the finite difference method), when it is necessary to satisfy the equilibrium equations inside the domain Ω. Thus, it is reasonable to use Lagrangians for formulations. Using the definition (5.117), we find that the following Lagrangians satisfy the disturbances (5.156) and (5.161):

$$L_2(v, \hat{p}^*) = -\int_\Omega \left[\frac{1}{2}A_{ijkl}\sigma_{kl}\sigma_{ij} - \sigma_{ij}\varepsilon_{ij}(v) + \rho F\cdot v\right] d\Omega - \int_{\Sigma_\sigma} P\cdot v\, d\Sigma, \tag{5.163}$$

$$L_3(v, \hat{p}^*) = \hat{L}_2(v, \hat{p}^*) + \int_{\Sigma_c} \sigma_N(-\delta_N + v_N)\, d\Sigma, \tag{5.164}$$

taking into account that $L_2(v, \hat{p}^*) = +\infty$ if $v \notin K$. The functional $L_2(v, \hat{p}^*)$ is determined by the formula (5.163) not only for $v \in K$, but everywhere. We notice that the use of the Lagrangian L_1, corresponding to the disturbance (5.153), does not lead to any new results.

According to Theorem 5.27, we state that the problem under consideration is equivalent to the following problems of computing the saddle point:

$$L_2(v, \hat{p}^*) \longrightarrow \sup_{\hat{p}^*\in\hat{Y}^*}\inf_{v\in V}, \tag{5.165}$$

$$L_3(v, p^*) \longrightarrow \sup_{p^*\in Y^*}\inf_{v\in V}, \tag{5.166}$$

where $\hat{Y}^*$ stands for the space of symmetric tensors of the second order. (Y^* is the space of the scalars on Σ_c.) If we put the homogeneous condition $u = g$ instead of the condition $u = 0$ on the boundary Σ_u of the substance Ω then, in some cases (in which the condition on Σ_u appears to be natural), there is an additional term in the expression for Φ^*:

$$\int_{\Sigma_u} \sigma_{ij}\nu_j g_i\, d\Sigma.$$

5.4.4 Generalization for the deformation theory of plasticity

Using the results of Sections 3.5 and 4.2.2, we find that the problem of one deformed body, yielding to the deformation theory of plasticity with a rigid stamp, is equivalent to the variational problem (see the formulae (3.132) and (3.133))

$$J(v) = J_0(v) - j(v) \longrightarrow \inf_{v \in K} . \tag{5.167}$$

It is clear that the disturbance (5.153) in this particular case leads to the problem $(\mathcal{P}_1^*)$, where the functional $J(v)$ is given with the formulae (3.132) and (3.133).

The disturbance of the Castigliano type (5.156) leads to the problem

$$\int_\Omega \left[-\frac{1}{2} A_{ijkl}\sigma_{kl}\sigma_{ij} + \frac{2}{E} \int_0^{\sigma_u(\hat\sigma)} \tilde\omega(\xi)\xi \, d\xi \right] d\Omega + \int_{\Sigma_c} \sigma_N \delta_N \, d\Sigma \longrightarrow \sup_{\hat\sigma \in M} , \tag{5.168}$$

where M is determined by the formula (5.160) as it was before, E is the Young modulus, $\sigma_u(\hat\sigma)$ is the stress intensity (see the definition (3.115)), and

$$\tilde\omega = \tilde\omega(\sigma_u) = \frac{1}{2\sigma_u} 3E e_u(\sigma_u) - (1-m). \tag{5.169}$$

Here m is the Poisson ratio and the function $e_u(\sigma_u)$ is the experimentally determined dependency of deformation intensity on the stress intensity (inverse to the dependency $\Phi(e_u)$, introduced by the formula (3.119)). The additional term with the function $\tilde\omega(\sigma_u)$ also emerges in the expressions for the functional $\Phi_3(0, p^*)$ and the Lagrangians L_2 and L_3.

Note that in the theory of the ideally plastic Hencky solids the governing equations contain

- The hypothesis about the additivity of elastic $\varepsilon_{ij}^e = A_{ijkl}\sigma_{kl}$ and plastic ε_{ij}^p deformations:
$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl} + \varepsilon_{ij}^p \tag{5.170}$$
- The yield condition (plasticity):
$$\Upsilon(\hat\sigma) \le 0, \tag{5.171}$$
where Υ is a continuous function of stresses
- The principle of the maximum plastic work [KH95]:
$$\varepsilon_{ij}^p(\tau_{ij} - \sigma_{ij}) \le 0 \quad \forall \tau_{ij} = \tau_{ji}, \ \Upsilon(\hat\tau) \le 0. \tag{5.172}$$

In order to get the variational principle, corresponding to the problem of computing the stresses during the contact of a deformed substance with a rigid smooth stamp, we integrate the inequality (5.172) in the domain Ω, substituting ε_{ij}^p by $\varepsilon_{ij} - A_{ijkl}\sigma_{kl}$. Using the Gauss formula and the equilibrium equations and the boundary conditions, we obtain

$$\int_{\Omega} A_{ijkl}\sigma_{kl}(\tau_{ij} - \sigma_{ij})\, d\Omega \geq \int_{\Sigma_c} (\tau_{ij} - \sigma_{ij}) v_i \nu_j \, d\Sigma. \tag{5.173}$$

(To simplify matters, we assume that $u|_{\Sigma_u} = 0$, mes $\Sigma_u \neq 0$.)

Applying the decomposition of the type (4.67) and using the condition of the absence of friction and the constraints on Σ_c

$$v_N \leq \delta_N, \quad \sigma_N \leq 0, \tag{5.174}$$

we state that

$$(\tau_{ij} - \sigma_{ij}) v_i \nu_j \geq (\tau_N - \delta_N)\delta_N \quad \forall \hat{\tau}. \tag{5.175}$$

Thus, this particular problem is equivalent to the variational inequality

$$\int_{\Omega} A_{ijkl}\sigma_{kl}(\tau_{ij} - \sigma_{ij})\, d\Omega \geq \int_{\Sigma_c} (\tau_N - \sigma_N)\delta_N \, d\Sigma, \tag{5.176}$$

which, in turn, determines the necessary and sufficient condition of the functional minimum

$$J(\hat{\tau}) = \int_{\Omega} \frac{1}{2} A_{ijkl}\tau_{kl}\tau_{ij}\, d\Omega - \int_{\Sigma_c} \delta_N \tau_N \, d\Sigma \tag{5.177}$$

on a set of functions M:

$$M = \{\hat{\tau} \mid \hat{\tau} \in L^2(\Omega);\ \operatorname{div}\hat{\tau} + \rho F = 0;\ \hat{\tau}\cdot\nu|_{\Sigma_\sigma} = P;$$
$$\tau_T|_{\Sigma_c} = 0;\ \tau_N|_{\Sigma_c} \leq 0;\ \Upsilon(\hat{\tau}) \leq 0\}. \tag{5.178}$$

This is a problem of type $(\mathcal{P}_2^*)$, but with the additional constraint on the stresses (5.171).

5.4.5 Contact problem for several elastic bodies

As was stated in Section 4.2.3, the problem

$$J(v) = \sum_{I=1}^{M} \left[\frac{1}{2} a^I(v^I, v^I) - L^I(v^I)\right] \tag{5.179}$$

is one of functional minimization on a set K, given by the formula (4.96). The Arrow–Hurwitz disturbance (5.153) leads to the following dual problem:

$$J(v) + \sum_I \int_{\Sigma_c^I} p_I^*(\delta_N - v_N^I + v_N^J)\, d\Sigma \longrightarrow \sup_{p_I^* \leq 0} \inf_{v \in V} \tag{5.180}$$

(see the nonpenetration condition (4.86)).

The computations using a disturbance of Castigliano type leads to a problem which only contains stresses:

$$\sum_I \left[-\frac{1}{2} \int_{\Omega^I} A^I_{ijkl} \sigma_{kl} \sigma_{ij} \, d\Omega + \int_{\Sigma^I_c} \sigma_N \delta_N \, d\Sigma \right] \longrightarrow \sup_{\hat{\sigma} \in M^I} , \tag{5.181}$$

where M^I is a set given by the formula (5.160) for the domain Ω^I with the boundary $\Sigma_I = \Sigma^I_\sigma \cup \Sigma^I_u \cup \Sigma^I_c$.

The combined disturbance of the type (5.161) results in the dual problem

$$\sum_I \left[-\frac{1}{2} \int_{\Omega^I} A^I_{ijkl} \sigma_{kl} \sigma_{ij} + \int_{\Sigma^I_c} (\delta_N - v^I_N - v^J_N) \delta_N \, d\Sigma \right] \longrightarrow \sup_{\hat{\sigma} \in M^I} \inf_{v \in V} . \tag{5.182}$$

(Here index J is the number of the bodies contacting to the domain Ω^I at a given point of the surface Σ^I_c.)

Analogously, the expression for the Lagrangians is formulated. The corresponding calculations are left as individual exercises.

5.4.6 Comments

The transformations, (5.28) and (5.29) is the Legendre transformation for the special problems (5.9)–(5.11). The definition of this transformation for an arbitrary dynamic system can be found in any textbook on analytical mechanics. As we know it is used in analytical mechanics (to obtain the canonical Hamilton equations, the Hamilton–Jacobi equations, etc.). It is also used in thermodynamics to build the full collection of thermodynamic potentials and to integrate ordinary differential equations by reducing them to the simplest form.

The Friedrichs transformation is described in more detail in [CH53]. The references of works devoted to the topic of applications of this transformation to the different mechanics problems are given throughout the text.

6

Nonstationary Problems and Thermodynamics

This chapter is devoted to an approaches to the problems in which we deal with the evolution of a system in time. We consider three types of evolution. The first one concerns the *dynamics* of the mechanical systems, i.e., wave propagation or vibrations. The second type of evolution problems are the *diffusion type problems*, e.g., heat conduction, diffusion of a substance through an another one (water through the soil, etc.). The third type is the static problem, but, in fact, we must investigate the dependence of the internal state on a varying external loads. An example of such a problem is the quasi-static plastic flow without viscosity (deformation of a perfectly plastic body).

All these problems can be formulated as the variational ones. If there are an unilateral constraints for an admissible fields (displacements, stresses, etc.) then we obtain variational or quasi-variational inequalities being the mathematical models of a system in evolution. Partly these models were considered in the previous chapters.

First, we formulate the classical differential principles of D'Alambert, Jourdain and Gauss which are, with additional hypotheses, the extremum conditions for a function or functional. Secondly, we formulate the classical and modern integral variational principles. The both can be used as a base for the mathematical modeling of a system with the unilateral constraints. In such a modeling we must add some additional hypotheses or principles, e.g., the Ostrogradski principle or nonnegativity of dissipation. The last one we use together with the Ziegler methods of irreversible forces to investigate the adhesion phenomena [Zie63].

6.1 Traditional principles and methods

6.1.1 Differential variational principles

It is well known that formally the dynamic equations for mechanical systems can be derived directly from the static equations through adding the inertia forces to the prescribed external forces. This method allows us to obtain the principle of virtual displacements in dynamics (the D'Alambert–Lagrange equation) directly from the equation (2.11)

$$\sum_{i=1}^{n}\left[\left(F_{ix}-m_i\frac{d^2x_i}{dt^2}\right)\delta x_i+\left(F_{iy}-m_i\frac{d^2y_i}{dt^2}\right)\delta y_i\right.$$
$$\left.+\left(F_{iz}-m_i\frac{d^2z_i}{dt^2}\right)\delta z_i\right]=0, \quad (6.1)$$

where m_i are the point masses and $r_i = (x_i, y_i, z_i)$ are their Cartesian coordinates ($i = 1, 2, \ldots, n$). If $F = \nabla U$ then the equation (6.1) can be rewritten as follows:

$$\sum_{i=1}^{n} m_i\ddot{r}_i\cdot\delta r_i=\delta U, \quad \ddot{r}_i\equiv\frac{d^2r_i}{dt^2}. \quad (6.2)$$

In the equations (6.1) and (6.2) the values δr_i are isochronous variations, which are equal to the difference of possible (virtual) values of coordinates $\tilde{r}_i = (\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)$ and the true values r_i at the time t: $\delta r_i = \tilde{r}_i - r_i$. Thus, the motion is considered to be a sequence of instantaneous states of the system equilibrium, and the "shaking" of the system from each such state through the isochronous change of coordinates leads to the zero work of all external forces, including the inertia forces. The equation (6.2), even for the potential forces, does not allow us to state that the instantaneous states of the system equilibrium correspond to stationary points of some function, because the left side of the equation (6.2) is not equivalent to the variation of a displacement function.

We have almost the same situation for the principle of virtual velocities (the Jourdain principle), which again considers the motion as a sequence of the equilibrium states, but the "shaking" of the system to find the equilibrium state is executed using all possible changes of velocities. Velocity variations are introduced as the differences between the admissible (virtual) velocities $\tilde{\dot{r}} = \dot{r}_i + \delta\dot{r}_i$ and the true velocities $\dot{r}_i$. The variations $\delta\dot{r}_i$ must be infinitesimally small. Notice that the coordinates r_i and accelerations $\ddot{r}_i$ stay unchanged. At instantaneous states of the system equilibrium the equations equilibrium

$$\sum_{i=1}^{n}\left[\left(F_{ix}-m_i\frac{d^2x_i}{dt^2}\right)\delta \dot{x}_i+\left(F_{iy}-m_i\frac{d^2y_i}{dt^2}\right)\delta \dot{y}_i\right.$$
$$\left.+\left(F_{iz}-m_i\frac{d^2z_i}{dt^2}\right)\delta \dot{z}_i\right]=0 \quad (6.3)$$

holds for all possible (compatible with the constraints) variations $\delta\dot{r}_i = (\delta\dot{x}_i, \delta\dot{y}_i, \delta\dot{z}_i)$.

Consider a (rather exotic) case when

$$F_{ix}=\frac{\partial U}{\partial \dot{x}_i}, \quad F_{iy}=\frac{\partial U}{\partial \dot{y}_i}, \quad F_{iz}=\frac{\partial U}{\partial \dot{z}_i} \quad (6.4)$$

with $\delta(\dot{x}_i^2) = 2\ddot{x}_i\dot{x}_i\delta t, \ldots$ (i.e., possible variations emerge only due to the time variation). Then the equation (6.3) is equivalent to the problem of computing the stationary point of the function

$$J(\dot{r}_1, \ldots, \dot{r}_n) = U(\dot{r}_1, \ldots, \dot{r}_n) - \frac{1}{2}\sum_{i=1}^{n} m_i|\dot{r}_i|^2. \tag{6.5}$$

The third well-known differential variational principle, the *Gauss principle*, views the motion as a sequence of the equilibrium states, but here, in order to choose the true instantaneous states, we vary the accelerations. Notice that the variation is isochronous (the coordinates, velocities, and time are not variable). For the true instantaneous equilibrium state, the following equation holds:

$$\sum_{i=1}^{n}\left[\left(F_{ix} - m_i\frac{d^2x_i}{dt^2}\right)\delta\ddot{x}_i + \left(F_{iy} - m_i\frac{d^2y_i}{dt^2}\right)\delta\ddot{y}_i + \left(F_{iz} - m_i\frac{d^2z_i}{dt^2}\right)\delta\ddot{z}_i\right] = 0. \tag{6.6}$$

Note that the rule for finding an admissible state allows the direct transition from the equation (6.6) to the minimization of the function

$$J(r_1, \ldots, r_n) = \sum_{i=1}^{n}\frac{1}{2m_i}(m_i\ddot{r}_i - F_i)^2. \tag{6.7}$$

This form of the problem is interpreted as a physical principle, the Gauss principle of least constraint. The last circumstance, the "physics" of the Gauss principle, leads to its successful use in various complicated problems, particularly, in the Ostrogradskii problem, considered in Sections 4.1 and 4.3. The Gauss principle, as is well known, applies to holonomic and nonholonomic problems, and the constraints are not necessarily linear with respect to the velocities.

6.1.2 Integral principles

The Hamilton principle is the most popular and practically used of all the integral variational principles of mechanics. To find the connection between the Hamilton principle and those developed in the preceding chapters, consider a system with one degree of freedom, described by the coordinate $q(t)$, the inertial characteristic m (= const) and the external force $F = F(t, q, \dot{q})$. Formulate the boundary value problem on the segment $t \in [t_1, t_2]$ for this system

$$-m\frac{d^2q}{dt^2} = -F, \tag{6.8}$$

$$q(t_1) = 0, \quad q(t_2) = 0. \tag{6.9}$$

Using standard reasoning as in Section 2.2, we can conclude that the problem (6.8) and (6.9) is equivalent to the variational equation

$$\int_{t_1}^{t_2} m \frac{dq}{dt}\frac{d\delta q}{dt}\, dt = -\int_{t_1}^{t_2} F\delta q\, dt \quad \forall \delta q = \tilde{q} - q, \tag{6.10}$$

where $q \in H_0^1(t_1,t_2)$, $\tilde{q} \in H_0^1(t_1,t_2)$ (see the definition of the space $H_0^1 \equiv W^{0,1}$ in Chapter 1). If the force F is potential, i.e., $F = -\partial V/\partial q$, then the theorem of potentiality (Section 3.4) allows the transition from the equation (6.10) to the problem of computing the stationary point of the functional

$$J(q) = \int_{t_1}^{t_2} (T - V)\, dt \equiv \int_{t_1}^{t_2} L\, dt, \quad q \in H_0^1(t_1,t_2), \tag{6.11}$$

where $T = m\dot{q}^2/2$ is the kinetic energy and $L = T - V$ is the Lagrange function (*action* by Hamilton). Notice that the stationary point of the functional (6.11) is the minimum only for sufficiently small time periods $t_2 - t_1$ (see [Ber83, Chapter III, § 3]). In the analysis of potentiality conditions for the operator, defined by the equation (6.10), we can extend the class of external effects, which allows the transition from the equation (6.10) to the problem (3.99).

The principle obtained in the problem of computing the stationary point of the functional (6.11) is called the *Hamilton principle.* If we want to generalize it for the system with several degrees of freedom we need, as before in the elementary problem, to fix the initial ($t = t_1$) and final ($t = t_2$) times, fix the system states

$$q_i^{(1)} = q_i(t_1), \quad q_i^{(2)} = q_i(t_2), \quad i = 1,2,\dots,n \tag{6.12}$$

at the initial and final times, compute the Hamiltonian $L = T - V$, and, finally, using, for example, the equation of the virtual displacement principle, prove the equation

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \tag{6.13}$$

being the Hamilton principle for arbitrary systems (with the prescribed forces).

Omitting the proof of the principle (6.13) (it can be found in almost every book on analytical mechanics), we introduce some important definitions. The value

$$p_r = \frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T}{\partial \dot{q}_r} \tag{6.14}$$

is called the *impulse.* As T is a quadratic form in the velocities, i.e.,

$$T = \frac{1}{2}\sum_{r,s} a_{rs}\dot{q}_r\dot{q}_s, \tag{6.15}$$

then

$$p_r = \sum_s a_{rs}\dot{q}_s. \tag{6.16}$$

Therefore,

$$2T = \sum_r p_r\dot{q}_r. \tag{6.17}$$

The value

$$H = \sum_r p_r\dot{q}_r - L \tag{6.18}$$

is called the *Hamilton function*. It can be demonstrated that $H = H(p, q)$. Thus

$$\begin{aligned}\delta\int_{t_1}^{t_2} L\,dt &= \int_{t_1}^{t_2}\sum_r\left(p_r\delta\dot{q}_r + \dot{q}_r\delta p_r - \frac{\partial H}{\partial q_r}\delta q_r - \frac{\partial H}{\partial p_r}\delta p_r\right)dt\\ &= \int_{t_1}^{t_2}\sum_r\left[\left(-\dot{p}_r - \frac{\partial H}{\partial q_r}\right)\delta q_r + \left(\dot{q}_r - \frac{\partial H}{\partial p_r}\right)\delta p_r\right]dt = 0. \end{aligned}\tag{6.19}$$

The equation (6.19) gives the canonical Hamilton equations

$$\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}. \tag{6.20}$$

6.1.3 Study of dissipative systems with the finite number of degrees of freedom

This section is devoted to the formal generalization of the Hamilton principle for systems with dissipation (see the monograph [MF53]). First, consider a system with one degree of freedom described by the equation

$$m\ddot{x} + R\dot{x} + kx = 0 \tag{6.21}$$

(a linear oscillator with friction). Consider, along with the system (6.21), the *mirror system* – an oscillator with negative friction, described by the variable x^*. Introducing the Lagrange function

$$L = m\dot{x}\dot{x}^* - \frac{1}{2}R(x^*\dot{x} - x\dot{x}^*) - kx^*x \tag{6.22}$$

and using the principle (6.21), we get the equations

$$m\ddot{x} + R\dot{x} + kx = 0, \quad m\ddot{x}^* - R\dot{x}^* + kx^* = 0 \tag{6.23}$$

for the original and mirror systems. The definition (6.22) leads to the expressions for the impulses

$$p = m\dot{x}^* - \frac{1}{2}Rx^*, \quad p^* = m\dot{x} + \frac{1}{2}Rx \tag{6.24}$$

and to the Hamilton function

$$H = \frac{1}{m}\left(p + \frac{1}{2}Rx^*\right)\left(p^* - \frac{1}{2}Rx\right) + kxx^*, \tag{6.25}$$

which is constant on the true trajectories, i.e., we formally get a conservative system.

The generalization for linear systems with many degrees of freedom is done easily. Introduce the kinetic energy

$$T = \frac{1}{2}\sum_{r,s} a_{rs}\dot{q}_r\dot{q}_s, \tag{6.26}$$

the potential energy

$$V = \frac{1}{2}\sum_{r,s} b_{rs}q_r q_s + \text{const} \tag{6.27}$$

and the force of resistance which depends linearly on the velocities

$$R_s = \sum_r R_{rs}\dot{q}_s. \tag{6.28}$$

Construct the Lagrange function for the system with double number of degrees of freedom (with coordinates $q_1, \ldots, q_n$, $q_1^*, \ldots, q_n^*$):

$$L = \sum_{r,s}\left[a_{rs}\dot{q}_r^*\dot{q}_s - \frac{1}{2}R_{rs}(q_r^*\dot{q}_s - \dot{q}_r^* q_s) - b_{rs}q_r q_s\right]. \tag{6.29}$$

Knowledge of this function solves (in principle) the problem of describing the dissipative system using a variational principle, but at the cost of doubling the number of degrees of freedom.

The Hamilton function is formulated with the same scheme that was used for a system with one degree of freedom.

6.1.4 Continuous conservative systems

Consider the wave equation for the finite segment $[0, l]$ with the homogeneous boundary conditions (fixed ends)

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \tag{6.30}$$

where c is the sound velocity $c = (T_0/\rho)^{1/2}$, T_0 is the string tension. Defining the kinetic energy

$$T = \frac{1}{2}\int_0^l \rho\left(\frac{\partial u}{\partial t}\right)^2 dx, \tag{6.31}$$

the potential energy

$$V = \frac{1}{2}\int_0^l \rho c^2 \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{6.32}$$

and the Hamiltonian

$$L = \frac{1}{2}\int_0^l \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 - c^2 \left(\frac{\partial u}{\partial x}\right)^2\right] dx, \tag{6.33}$$

we find that the boundary value problem for the equation (6.30) is equivalent to the problem of computing the stationary point of the functional

$$\mathcal{L}(v) = \int_{t_1}^{t_2} L\,dt \tag{6.34}$$

on the set of functions satisfying the given boundary conditions, at the points $t = t_1$, $t = t_2$, $x = 0$, $x = l$. This principle has no great practical significance, because, in practice, we must usually solve the Cauchy problem – problem with the initial conditions. Using known methods, we can formally reduce the Cauchy problem to a boundary problem if at the point $t = t_2$ we assign the solution value corresponding to the initial conditions of the problem (beforehand unknown). The problems emerging in this procedure do not allow us to use all the advantages of the variational statement.

If the considered system is under load with linear density F and if it is on the linear Winkler support (foundation), then we get the following equation instead of the equation (4.27):

$$\rho\frac{\partial^2 u}{\partial t^2} - T_0\frac{\partial^2 u}{\partial x^2} = F - ku. \tag{6.35}$$

The Lagrange function takes the form

$$L = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - T_0\left(\frac{\partial u}{\partial x}\right)^2 - ku^2 + 2Fu\right] dx. \tag{6.36}$$

Consider the general case, when we have some vector field u with m components, each of those is a function of n independent variables $x_1, \dots, x_n$, one of which is the time. Suppose that the operator corresponding to this system satisfies the conditions of potentiality. With this hypothesis we obtain the variational principle

$$\delta\mathcal{L} = \delta\int\cdots\int L(u, \nabla u, x)dx_1 \dots dx_n = 0. \tag{6.37}$$

Introduce a Cartesian coordinate system and decompositions $u = u_r k_r$, $\nabla u = u_{rs}k_r k_s = (\partial u_r/\partial x_s)k_r k_s$. It is easy to get the following Euler equations:

$$\sum_{s=1}^n \frac{\partial}{\partial x_s}\left(\frac{\partial L}{\partial u_{rs}}\right) = \frac{\partial L}{\partial u_r}, \quad r = 1, 2, \dots, m. \tag{6.38}$$

In fact, first we have the system (6.38), and then we construct the variational principle (6.37).

Consider as an example the dynamic problem for the linear theory of elasticity. In this theory the displacement vector u satisfies the following system of equations:

$$-\frac{\partial}{\partial x_j}(a_{ijkl}\varepsilon_{kl}(u)) = \rho F_i - \rho\partial^2 u_i/\partial t^2. \tag{6.39}$$

This is the system of the Euler equations for the functional

$$\mathcal{L}(v) = \int_{t_1}^{t_2}\int_{\Omega} L(v, \nabla u)\, d\Omega\, dt, \tag{6.40}$$

where

$$L = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\hat{\sigma}(u)\cdot\cdot\hat{\varepsilon}(u) - \rho F\cdot u \tag{6.41}$$

is the density of the Lagrange function. The stresses $\hat{\sigma}(u)$ are given by the Hooke law, strains $\hat{\varepsilon}(u)$ are given by the Cauchy formulae (2.98).

6.1.5 Example of a dissipative continuous system

Consider the heat conduction equation

$$\Delta u = a^2\frac{\partial u}{\partial t}, \quad x\in\Omega, \tag{6.42}$$

with the zero boundary conditions on the boundary $\Sigma = \partial\Omega$ of the domain Ω. Introduce an additional equation with the mirror temperature u^* (as it was done in Section 6.1.3), and compose the Lagrange function

$$L(u,u^*) = -\nabla u\cdot\nabla u^* - \frac{1}{2}a^2\left(u^*\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial t}\right). \tag{6.43}$$

The corresponding variational principle

$$\delta\int_{t_1}^{t_2}\int_{\Omega} L\, d\Omega\, dt = 0 \tag{6.44}$$

gives the equation (6.42) and the equation

$$\Delta u^* = -a^2\frac{\partial u}{\partial t} \tag{6.45}$$

for the mirror temperature u^*.

6.2 Gurtin method

6.2.1 Wave equation

The method, introduced by Gurtin [Gur64a, Gur64b], can be applied (as opposed to the Hamilton principle) to the Cauchy problems, where the initial conditions are formulated in time, and to the dissipative systems without duplication of the number of the unknown functions. We illustrate this method with the example of the initial-boundary problem for the wave equation

$$\varDelta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad x \in \varOmega \subset \mathbb{R}^n, \tag{6.46}$$

$$u(x,0) = u_0(x), \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = u_1(x), \quad x \in \varOmega, \tag{6.47}$$

$$u|_{\varSigma} = U(x,t), \quad x \in \varSigma,\ t \in [0,+\infty), \tag{6.48}$$

where c is the sound speed and u_0, u_1, U are given functions. We apply the Laplace transformation in time to the equation (6.46), taking into account that the function $u(x,t)$ must satisfy the conditions (6.47) at $t = 0$. Recall that the *Laplace transform of the real variable function* $f(t)$ is the function $f^*(p)$ of the complex variable p

$$f^*(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{6.49}$$

For a function $f(t)$ differentiable in the segment $(0,+\infty)$ the following formula holds:

$$\int_0^\infty e^{-pt}\frac{df}{dt}\, dt = pf^*(p) - f(0). \tag{6.50}$$

For twice differentiable function $f(t)$ on the segment $(0,+\infty)$ the following formula holds:

$$\int_0^\infty e^{-pt}\frac{d^2 f}{dt^2}\, dt = p^2 f^*(p) - pf(0) - f'(0). \tag{6.51}$$

Suppose the abscissa of convergence (i.e., the upper edge of the numbers $Re(p)$, for which the integral (6.49) converges) is equal to γ_c, and let $\alpha > \max\{\gamma_c, 0\}$. Then

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} f^*(p)e^{pt}dp = \begin{cases} f(t), & t > 0,\\ 0, & t < 0.\end{cases} \tag{6.52}$$

This formula is called the *Mellin inversion formula*. (The proofs of these statements can be found in any textbook on operational calculus, e.g., in [Doe74].)

As the result of applying the transformation (6.49) to the equation (6.46) according to the formula (6.51), we find that

$$c^2 \Delta u^*(x,p) - p^2 u^*(x,p) = -pu_0(x) - u_1(x). \tag{6.53}$$

Dividing the left- and right-hand parts of this equation by p^2 and applying the Mellin inversion formula (6.52) to the left- and right-hand parts of the equation, we get

$$g * \Delta u - u = -u_0(x) - tu_1(x) \equiv f(x,t). \tag{6.54}$$

In the equation (6.54) the “asterisk” means the convolution operation:

$$v(x,t) * w(x,t) = \int_0^t v(x,t-\tau) w(x,\tau)\, d\tau. \tag{6.55}$$

Besides, we used the formula:

$$g(x,t) \equiv c^2 t \tag{6.56}$$

and the inversion (6.52).

Thus, any solution of the problems (6.46) and (6.47) (the conditions (6.48) on the boundary Σ are not fixed yet) satisfies the integro-differentiated equation (6.54). Inversely, any solution of the equation (6.54) satisfies the equation (6.46) and the conditions (6.47). For the proof it is enough to reapply the transformation (6.49), use the theorem of convolution [Doe74]

$$\int_0^\infty (v * w) e^{-pt}\, dt = v^* w^*, \tag{6.57}$$

and perform further computations in the order inverse to those which led to the equation (6.54).

The Gurtin variational principle is obtained through considering the problems, (6.54) and (6.48) as a problem with the parameter t, and replacing ordinary multiplication by the convolution multiplication (6.55). The theoretical foundations of the algebra of convolution multiplication were developed in the works of Mikusinski [MB87]. Gurtin's main result is formulated in the form of a theorem:

Theorem 6.1. *Let K be the set of functions, satisfying the boundary condition (6.48). Then the boundary problems (6.54) and (6.48) is equivalent to the problem of computing the stationary point of the functional*

$$\Lambda_t(v) = \frac{1}{2} \int_\Omega [v * v + g * \nabla v * \nabla v + 2f * v]\, d\Omega \tag{6.58}$$

on the set K. The function g is given by the formula (6.56).

Proof. We determine the element

$$v = u + \varepsilon\delta u, \tag{6.59}$$

where δu is a variation of the stationary point u and ε is arbitrary. From the random choice of ε it follows that

$$\delta u = 0 \quad \text{on the set } \Sigma \times (0, +\infty). \tag{6.60}$$

Using the associativity and commutativity properties of the convolution and the Gauss theorem, we state that the variation $\delta\Lambda_t(u)$ of the functional Λ_t on K is

$$\delta\Lambda_t = \int_\Omega [u - g * \Delta u + f] * \delta u(x,t)\, d\Omega \quad \forall\delta u,\ \delta u = 0 \text{ on } \Sigma \times (0,+\infty). \tag{6.61}$$

Let u be the solution of the initial problems (6.46)–(6.48) and, therefore, of the problems (6.54) and (6.48). The equality (6.61) implies that

$$\delta\Lambda_t(u) = 0, \quad u \in K,\ \delta u = 0 \text{ on } \Sigma \times (0,+\infty), \tag{6.62}$$

i.e., u is the stationary point of the functional (6.58). Inversely, let u be the stationary point of the functional Λ_t on the set K. Then, taking into account the expression (6.61), the equation (6.62) leads to the following:

$$\int_\Omega \int_0^t [u - q * \Delta u + f](x, t-\tau)\delta u(x,\tau)\, d\tau\, d\Omega = 0 \qquad \forall\delta u,\ \delta u = 0 \text{ on } \Sigma \times (0,+\infty). \tag{6.63}$$

We notice that the equation (6.63) is derived from the equation (6.54). This fact is outwardly evident and its strong proof can be found in the Titchmarsh theorem (see [Tit48, p. 66, 190]), which we omit here due to its awkwardness.

6.2.2 Heat conduction equation

Consider the problem

$$a^2\Delta u = \frac{\partial u}{\partial t}, \qquad x \in \Omega \subset \mathbb{R}^n,\ t \in (0,+\infty), \tag{6.64}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{6.65}$$

$$u(x,t) = U(x,t), \quad x \in \Sigma = \partial\Omega,\ t \in (0,+\infty), \tag{6.66}$$

where $u = u(x,t)$ is the temperature in the domain Ω, a is the constant given by the thermodynamic properties of the material, and u_0, U are given functions.

The procedure of Section 6.2.1 is easily adapted to the problems (6.64)–(6.66). The application of the Laplace transformation (6.49), taking into account the formula (6.50) and the inversion formulae, leads to the equation

$$q * \Delta u - u = r, \tag{6.67}$$

where

$$q = a^2 = \text{const}, \quad r = -u_0(x). \tag{6.68}$$

The equation (6.67) is equivalent to the equations (6.64) and (6.65). We can prove this statement in exactly the same way as we proved the equivalence of the equation (6.54) to the equations (6.46) and (6.47). Then, we introduce the functional

$$M_t(v) = \frac{1}{2} \int_{\Omega} [v * v + q * \nabla v * \nabla v + 2r * v] \, d\Omega \tag{6.69}$$

and prove the following theorem:

Theorem 6.2. *Let K be the set of functions satisfying the boundary condition (6.66). Then the boundary problem (6.64)–(6.66) is equivalent to the problem of computing the stationary point of the functional $M_t(v)$ on the set K.*

The proof is the same as in Theorem 6.1.

6.2.3 Dynamic problem of the linear theory of elasticity

We summarize the results obtained above for the case of vector fields. We consider one general summary – the initial-boundary value problem for a linearly elastic material, see equations (2.113)–(2.118). All computations and reasonings are valid if the following assumptions hold [Gur64a]:

1. The density $\rho > 0$ and is continuously differentiable in the domain Ω
2. The tensors of the elasticity modulus a_{ijkl} and the compliance modulus A_{ijkl} are continuously differentiable in the domain Ω
3. The initial states $u_0(x)$ and the initial velocities $u_1(x)$ are continuous in the domain Ω
4. The displacements g given on the segment of the boundary Σ_u are continuous on the variety $\Sigma_u \times [0, +\infty)$
5. The density of the surface tractions P given on the segment of Σ_σ is a piece-wise differentiable function on $\Sigma_\sigma \times [0, +\infty)$

The *state* S of the elastic body in the domain Ω refers to the set of three fields, the displacements field u, the deformation field $\hat{\varepsilon}$ and the stress field $\hat{\sigma}$: $S = \{u, \hat{\varepsilon}, \hat{\sigma}\}$. We call this state *admissible* if

$$u_i \in C^{1,2}, \quad \varepsilon_{ij} \in C^{0,0}, \quad \sigma_{ij} \in C^{1,0}, \tag{6.70}$$

where $C^{M,N}$ are the sets of functions of independent variables $x_1, \ldots, x_n$, t, and the derivatives

$$\frac{\partial^{m+k} f(x,t)}{\partial x_{i_1} \cdots \partial x_{i_n} \partial t^k} \equiv f^k_{i_1 \cdots i_n}, \quad i_1 + \ldots + i_n = m,$$
$$m = 0, 1, \ldots, M, \;\; k = 0, 1, \ldots, N \tag{6.71}$$

are continuous on $\bar{\Omega} \times [0, +\infty)$.

At the first stage we apply the Laplace transformation in t to the motion equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \Phi_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad \Phi_i \equiv \rho F_i. \tag{6.72}$$

The result is formulated in the form of the following theorem:

Theorem 6.3. *Let $u_i \in C^{0,2}, \sigma_{ij} \in C^{1,0}$. Then the functions u_i, σ_{ij} satisfy the motion equations (6.72) and the initial conditions (2.118) if and only if*

$$g * \sigma_{ij,j} + f_i = \rho u_i, \quad (x,t) \in \Omega \times [0, +\infty], \tag{6.73}$$

where

$$g = g(t) = t, \tag{6.74}$$

$$f_i(x,t) = [g * \Phi_i](x,t) + \rho(x)[t u_{1i}(x) + u_{0i}(x)]. \tag{6.75}$$

$u_{1i}(x) + u_{0i}(x)$ are the Cartesian component of the initial data in (2.118).

The proof repeats the reasonings which led from the equation (6.46) to the equation (6.54), with the same definition of convolution (6.55). Incorporating the boundary conditions

$$u|_{\Sigma_u} = U, \quad \hat{\sigma} \cdot \nu|_{\Sigma_\sigma} = P, \tag{6.76}$$

into the equation (6.73) with the Hooke law (2.118) and the Cauchy formulae and interlinking $\hat{\varepsilon}$ and u, we obtain the solution of the initial boundary value problem (2.113)–(2.118).

Further reading concerns building the analogues of the Lagrange, Castigliano, Hu–Washizu, and Reissner variational principles, which were formulated earlier for static problems (see Section 5.2). The most difficult task is building the variational principles in the stresses (which is, as explained later, the analogue of the Castigliano principle). All the others can be obtained through the use of the equation (6.73) in combination with different additional constraints of the group (2.113)–(2.118), the symmetry conditions

$$\sigma_{ij} = \sigma_{ji}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \tag{6.77}$$

and the Hooke law in the following two forms:

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}, \quad \varepsilon_{ij} = A_{ijkl}\sigma_{kl}, \quad a_{ijkl}A_{klmn} = \delta_{im}\delta_{jn}. \tag{6.78}$$

Before we formulate these variational principles, we describe the Ignaczak theorem [Ign63], which solves the problem concerning the investigation of the variational principle in the stresses.

Theorem 6.4. *Let $\sigma_{ij} \in C^{2,0}$ and the symmetry conditions (6.77) be satisfied. Then the stresses field σ_{ij} is the solution of the given problem if and only if*

$$\hat{\varepsilon}[(g' * \operatorname{div} \hat{\sigma}) + f'] = {}^4\hat{A} \cdot \cdot \hat{\sigma} \tag{6.79}$$

in the domain $\Omega \times [0, +\infty)$ and

$$g' * \operatorname{div} \hat{\sigma} = U - f', \qquad (x,t) \in \Sigma_u \times [0, +\infty] \tag{6.80}$$

$$\hat{\sigma} \cdot \nu = P, \qquad (x,t) \in \Sigma_\sigma \times [0, +\infty], \tag{6.81}$$

$$\hat{\varepsilon}(\varphi) = \frac{1}{2}(\nabla\varphi + \nabla\varphi^T), \tag{6.82}$$

$$g'(x,t) = \rho^{-1}(x) g(t), \tag{6.83}$$

$$f'(x,t) = \rho^{-1}(x) f(x,t). \tag{6.84}$$

Proof. Let $\hat{\sigma}$, $\hat{\varepsilon}$, u be the solution of the initial problem. According to Theorem 6.3, the fields u and s are interlinked by the equation (6.73). Therefore,

$$u_i = \frac{1}{\rho} g * \sigma_{ij,j} + \frac{1}{\rho} f_i = g' * \sigma_{ij,j} + f_i'. \tag{6.85}$$

Computing the deformation tensor $\hat{\varepsilon}$ on the field u by the formula of (6.82) and deploying the second formula of (6.78), we get the equation (6.79). Doing the computations and the reasonings in the inverse order, we go from the equation (6.79) (with the conditions (6.80)–(6.84)) to the initial problem. Thus the proof of the theorem is completed.

Now we discuss building the variational principles. We start with the most general case (of the Hu–Washizu type, see Section 5.2, Problem 5.14) from which all other principles can be derived through applying additional constraints. The Gurtin fundamental result is here formulated in the form of the theorem.

Theorem 6.5. *The problem (2.113)–(2.118) is equivalent to the problem of computing the stationary point of the functional*

$$\begin{aligned}
\Lambda_t(S) = \frac{1}{2} \int_\Omega & \{a_{ijkl}(x)[g * \varepsilon_{ij} * \varepsilon_{kl}](x,t) + \rho(x)[u_i * u_i](x,t) \\
& -2[g * \sigma_{ij} * \varepsilon_{ij}](x,t) - 2[(g * \sigma_{ij,j} + f_i) * u_i](x,t)\}\, d\Omega \\
& + \int_{\Sigma_u} [g * (\sigma_{ij}\nu_j) * U_i](x,t)\, d\Sigma + \int_{\Sigma_\sigma} [g * (\sigma_{ij}\nu_j - P_i) * u_i](x,t)\, d\Sigma
\end{aligned} \tag{6.86}$$

on the set K of the feasible fields $\{u, \hat{\varepsilon}, \hat{\sigma}\}$.

Proof. We compute the variation $\delta\Lambda_t$ (varying independently the fields u, $\hat{\varepsilon}$, $\hat{\sigma}$):

$$\delta\Lambda_t = \int_\Omega \{[g * (a_{ijkl}\varepsilon_{kl} - \sigma_{ij}) * \delta\varepsilon_{ij}] - [(g * \sigma_{ij,j} + f_i - \rho u_i) * \delta u_i]$$
$$+ g * [0{,}5(u_{i,j} + u_{j,i}) - \varepsilon_{ij}] * \delta\sigma_{ij}\} \, d\Omega$$
$$+ \int_{\Sigma_u} [g * (U_i - u_i) * (\delta\sigma_{ij}\nu_i)] \, d\Sigma + \int_{\Sigma_\sigma} [g * (\sigma_{ij}\nu_j - P_i) * \delta u_i] \, d\Sigma. \tag{6.87}$$

To obtain the formula (6.87), we use the Gauss formula. If the set of fields $S = \{u, \hat{\varepsilon}, \hat{\sigma}\}$ is the solution of the initial problem, then equation (6.73) implies

$$\delta\Lambda_t = 0. \tag{6.88}$$

If now the equation (6.88) holds on some element $\{u, \hat{\varepsilon}, \hat{\sigma}\}$, then, using the assumption about the independence of the variations δu, $\delta\hat{\varepsilon}$, $\delta\hat{\sigma}$ and the Titchmarsh theorem (see Section 6.1.2 and [Tit48]), we find that all the equations and conditions of the initial problem are satisfied. The theorem is proved.

Now we apply the Cauchy dependence (2.115) to the problem of computing the stationary point of the functional Λ_t (i.e., we eliminate the deformation field $\hat{\varepsilon}$). Then, we get the analogue of the Hellinger–Reissner variational principle: the problem of computing the stationary point of the functional

$$M_t(\hat{\sigma}, u) = \int_\Omega \left\{ g * \sigma_{ij} * \varepsilon_{ij} - \frac{1}{2} A_{ijkl}[g * (\sigma_{ij} * \sigma_{kl})] \right.$$
$$\left. + \frac{1}{2}\rho u_i * u_i - f_i * u_i \right\} d\Omega - \int_{\Sigma_u} [g * (\sigma_{ij}\nu_i) * (u_i - U_i)] \, d\Sigma$$
$$- \int_{\Sigma_\sigma} (g * P_i * u_i) \, d\Sigma \tag{6.89}$$

on the set of admissible fields K, is equivalent to the initial problem (2.113)–(2.118). The proof is accomplished in the same way as in Theorem 6.5.

Using the Cauchy dependence and the Hooke law, we eliminate the fields $\hat{\sigma}$ and $\hat{\varepsilon}$, and thus we get the analogue of the Lagrange variational principle, according to which the problem (2.113)–(2.118) is equivalent to the problem of the stationary point of the functional

$$J_t(u) = \frac{1}{2} \int_\Omega \{a_{ijkl}[g * u_{i,j} * u_{k,l}] + \rho u_i * u_i - f_i * u_i\} \, d\Omega - \int_{\Sigma_\sigma} (g * P_i * u_i) \, d\Sigma \tag{6.90}$$

on the set of all admissible displacement fields u. The permissibility requirement now contains the differentiability on x, and it must satisfy the kinematic boundary condition, the first condition of (6.76).

Finally, using the Ignachak theorem (6.24), we obtain the variational principle in the stresses: the problem (2.113)–(2.118) is equivalent to the problem of computing the stationary point of the functional

$$
\begin{aligned}
J_t^*(\hat{\sigma}) = & \frac{1}{2} \int_\Omega \left\{ (g' * \sigma_{im,m} * \sigma_{ij,j}) + A_{ijkl}(\sigma_{ij} * \sigma_{kl}) - (f'_{i,j} * \sigma_{ij}) \right\} \, d\Omega \\
& + \int_{\Sigma_u} [(f'_i - U_i) * (\sigma_{ij}\nu_i)] \, d\Sigma \\
& + \int_{\Sigma_\sigma} [g' * (P_i - \sigma_{ij}\nu_j) * (\sigma_{im,m})] \, d\Sigma
\end{aligned} \tag{6.91}
$$

on the set of the admissible stress fields $\hat{\sigma}$. For the proof, we compute the variation δJ_t^* and use the equations (6.79)–(6.84).

This principle is not the analogue of the Castigliano principle. For statics there is no analogue of the formula (5.85) for the elimination of displacements.

6.2.4 Theory of linear viscoelasticity

The procedure is generalized for the problems in the theory of viscoelasticity [Chr71]. The fundamental relations of the theory of viscoelasticity – particularly that for investigating the construction of the polymeric materials – are written in different forms, see, e.g., [IP70, KMU85].

To formulate the variational principles, the realized relations are written as the Stieltjes convolutions in time:

$$
\sigma_{ij} - G_{ijkl} * d\varepsilon_{kl} = \int_0^t G_{ijkl}(t-\tau) \frac{\partial \varepsilon_{kl}(\tau)}{\partial \tau} \, d\tau, \tag{6.92}
$$

$$
\varepsilon_{ij} = J_{ijkl} * d\sigma_{kl} = \int_0^t J_{ijkl}(t-\tau) \frac{\partial \sigma_{kl}(\tau)}{\partial \tau} \, d\tau, \tag{6.93}
$$

where G_{ijkl}, J_{ijkl} are the given functions.

To keep things simple, we consider quasi-static problems, where forces of inertia are neglected in the motion equations

$$
\sigma_{ij,i} + \Phi_i = 0. \tag{6.94}
$$

(The dynamics problems are obtained by applying the procedure of Section 6.2.3.)

The analogue of the Hu–Washizu variational principle is formulated in the following theorem.

Theorem 6.6. *The boundary problem for the equation (6.94), into which we incorporate the relations (6.92) or (6.93), the Cauchy equations and the boundary conditions (6.76), is equivalent to the problem of computing the stationary point of the functional*

$$
\begin{aligned}
\Lambda_t(u, \hat{\varepsilon}, \hat{\sigma}) = & \int_\Omega \left[\frac{1}{2} G_{ijkl} * d\varepsilon_{ij} * d\varepsilon_{kl} - \sigma_{ij} * d\varepsilon_{ij} - (\sigma_{ij,j} + \Phi_i) * du_i \right] d\Omega \\
& + \int_{\Sigma_u} [(\sigma_{ij}\nu_i) * dU_i] \, d\Sigma + \int_{\Sigma_\sigma} [(\sigma_{ij}\nu_i - P_i) * du_i] \, d\Omega.
\end{aligned} \tag{6.95}
$$

Proof. The functional (6.95) is computed and, using the Gauss theorem, is brought to the following form:

$$\delta\Lambda_t = \int_\Omega [(G_{ijkl} * d\varepsilon_{kl} - \sigma_{ij}) * d\delta\varepsilon_{ij} - (\sigma_{ij,j} + \Phi_i) * d\delta u_i$$
$$- (\varepsilon_{ij} - 0{,}5u_{i,j} - 0{,}5u_{j,i}) * d\delta\sigma_{ij}]\, d\Omega + \int_{\Sigma_\sigma} [(\sigma_{ij}\nu_j - P_i) * d\delta u_i]\, d\Sigma$$
$$+ \int_{\Sigma_u} [(U_i - u_i) * d\delta\sigma_i]\, d\Sigma, \quad \sigma_i \equiv \sigma_{ij}\nu_j, \tag{6.96}$$

where δu_i, $\delta\varepsilon_{ij}$, $\delta\sigma_{ij}$ are independent variations. If the point $(u, \hat{\varepsilon}, \hat{\sigma})$ is the solution of the initial problem, then it follows from the formula (6.96) that

$$\delta\Lambda_t(u, \hat{\varepsilon}, \hat{\sigma}) = 0. \tag{6.97}$$

Inversely, if the equation (6.97) is satisfied, then all the equations and the conditions of the initial problem are derived from the independence of δu, $\delta\hat{\varepsilon}$, $\delta\hat{\sigma}$, and the equation (6.96).

Other variational principles can be obtained as in the previous paragraph. For example, adding the Cauchy relation to the problem of computing the stationary point of the functional (6.95) (i.e., eliminating the deformation field $\hat{\varepsilon}$), we obtain the analogue of the Hellinger–Reissner variational principle: to compute the stationary point of the functional

$$M_t(\sigma_u) = \int_\Omega \left[\frac{1}{2} J_{ijkl} * d\sigma_{ij} * d\sigma_{kl} - \frac{1}{2}\sigma_{ij} * d(u_{i,j} + u_{j,i}) + \Phi_i * du_i\right] d\Omega$$
$$+ \int_{\Sigma_u} [\sigma_i * d(u_i - U_i)]\, d\Sigma + \int_{\Sigma_\sigma} (P_i * du_i)\, d\Sigma. \tag{6.98}$$

Using the law (6.92) and the Cauchy equations, we eliminate the stress field $\hat{\sigma}$ and get the analogue of the Lagrange principle (the minimum of the potential energy): to find the stationary point of the functional

$$J_t(u) = \int_\Omega \left[\frac{1}{2} G_{ijkl} * d\varepsilon_{ij} * d\varepsilon_{kl} - \Phi_i * du_i\right] d\Omega - \int_{\Sigma_\sigma} (P_i * du_i)\, d\Sigma \tag{6.99}$$

on the set of admissible displacements fields u.

Eliminating the deformations in (6.98) with the law (6.93) and requiring that the feasible stress fields satisfy the equilibrium equations (6.94) and the power boundary conditions on Σ_σ, we get the analogue of the Castigliano principle: to find the stationary point of the functional

$$J_t^*(\hat{\sigma}) = \frac{1}{2}\int_\Omega (J_{ijkl} * d\sigma_{ij} * d\sigma_{kl})\, d\Omega - \int_{\Sigma_u} [(\sigma_{ij}\nu_j) * dU_i]\, d\Sigma \tag{6.100}$$

on the set of stress fields, satisfying the given constraints. The described results are also due to Gurtin [Gur63].

To conclude this paragraph, we notice that the variational principles about the minimum of the functional are set for linear-viscoelastic materials and for the special forms of the processes of changing stresses and deformations – with separation of the temporal variable, see [Chr71]. Note that neither the classic variational principles for the nonstationary problems nor the Gurtin theorem lead to problems of functional minimization.

6.3 Thermodynamics and mechanics of the deformed solids

This section is devoted to some problems of a continuum deformation with the variable temperature accompanied by increase of entropy. This approach allows us to construct the mathematical models of a physical and mechanical phenomenon which does not contradict to the laws of thermodynamics. We use the definitions of entropy and other thermodynamic functions given in Section 1.5. The difference consists of dependencies of all the thermodynamical parameters on the spatial coordinates, i.e., we deal with fields instead of numerical variables.

6.3.1 Generalities

First law of thermodynamics

Recall that in Chapter 1 we used the so-called *localization principle* with which we investigated an infinitesimal neighborhood of a given point in a *heterogeneous* (nonhomogeneous) system and supposed that any state of such neighborhood is homogeneous. We now consider a finite domain Ω, and introduce the densities defined in this domain denoted by the small letters – material density ρ, total energy e and kinetic energy k per unit mass, etc. (excluding the temperature T and mass forces F). We consider a process composed by an infinitesimal increase of the thermodynamical parameters $\{\pi_0, \pi_1, \ldots, \pi_m\} = \{\pi_i\}_{i=0}^m \equiv \mathcal{E}$ (see Chapter 1) being the densities of the corresponding fields. By definition,

$$\delta E = \int_\Omega \rho \delta e \, d\Omega = \int_\Omega \rho(\delta k + \delta u) \, d\Omega \tag{6.101}$$

$$k = \frac{1}{2}\rho v^2, \quad \delta K = \int_\Omega \rho \delta k \, d\Omega, \quad \delta U = \int_\Omega \rho \delta u \, d\Omega,$$

where u is the density of the internal energy and v is the module of the particle velocity. Other densities useful in applications will be introduced later.

Let Ω be a domain in $\mathbb{R}^3$ which contains a material substance (e.g., a deformed solid). Consider an arbitrary subdomain $\Omega_1 \subset \Omega$ as a thermodynamic system. It follows from the first law of thermodynamics for this system that $\delta E = dE$. By supposition, dE is the sum of three terms,

$$dE = \delta A^e + \delta Q^e + \delta Q^*, \tag{6.102}$$

where δA^e is the inflow of the mechanical energy, i.e., the work of mass and surface forces, δQ^e is the heat inflow, and δQ^* corresponds to all other energy inflows which differ from the mechanical and heat ones. We use the notation δ for an increment. Note that, in general, none of the terms of (6.102) is totally differential.

Suppose that Ω is a deformed body and that

$$\delta A^e = \int_{\Sigma_1} (\hat{\sigma} \cdot \nu) \cdot v \, dt \, d\Sigma + \int_{\Omega} \rho F \cdot v \, dt \, d\Omega, \tag{6.103}$$

where $\hat{\sigma}$ is the stress tensor, ν is the outward drawn unit vector orthogonal to the surface $\Sigma_1 = \partial\Omega_1$, $v\,dt$ is the increment of the particle displacement in Ω_1 which belongs to the set of the thermodynamical parameters, and F is the density of the mass forces.

To exclude the kinetic energy in the first law of thermodynamics, we use the law of the mechanical energy conservation. For this we consider equation of motion (2.103) written in the following form:

$$\nabla \cdot \hat{\sigma} + \rho F = \rho \frac{\partial v}{\partial t}. \tag{6.104}$$

Calculate the inner product of the equation (6.104) with the velocity vector v and integrate over the domain Ω_1. Using the definition of the kinetic energy and the Gauss formula, we obtain:

$$-\int_{\Omega_1} \hat{\sigma} \cdot \cdot d\hat{\varepsilon} \, d\Omega + \int_{\Sigma} (\hat{\sigma} \cdot \nu) \cdot v \, dt \, d\Sigma + \int_{\Omega} \rho F \cdot v \, dt \, d\Omega = \int_{\Omega_1} \rho dk \, d\Omega, \tag{6.105}$$

where $\hat{v}\,dt = d\hat{\varepsilon}(u)$, $\hat{\varepsilon}(u)$ is the increment of the Cauchy strain tensor (2.98). Note that all the calculations can be generalized to an arbitrary deformation process. (In this section we investigate the processes in a linear elastic body only.) The equality (6.105) is the *law of conservation of the mechanical energy* (or the *theorem on the living forces*).

Define the densities

$$\delta Q^e = \int_{\Omega_1} \delta q^e \, d\Omega, \quad \delta Q^* = \int_{\Omega_1} \delta q^* \, d\Omega \tag{6.106}$$

of heat inflow δq^e and inflow δq^* which is used for the modeling of a complex processes characterized by the energy Q^* (see the definition (6.102)).

Using the definitions (6.101) and (6.106), the equality (6.105), the law (6.102), and the arbitrariness of the domain $\Omega_1 \in \Omega$, we obtain the so-called *equation of the heat inflow*:

$$\rho du = \hat{\sigma} \cdot \cdot d\hat{\varepsilon} + \delta q^e + \delta q^*. \tag{6.107}$$

If we now introduce additional hypotheses on the internal energy u, on the inflows δq^e and δq^*, and on the governing equations, we can construct a mathematical model of the deformation processes accompanied by heating, fracture, phase transition, etc.

Second law of thermodynamics

We now formulate the main results obtained in Chapter 1 with the second law using the definitions of densities. Note first that all the statement of Section 1.5.3 are valid for the densities in a continuum. For example, if we introduce the density s of entropy as

$$S = \int_{\Omega_1} \rho s \, d\Omega \tag{6.108}$$

then it follows from the second law that in a reversible process the following equality holds:

$$dS = \int_{\Omega_1} \rho ds \, d\Omega = \int_{\Omega_1} \frac{\delta q^e}{T} \, d\Omega. \tag{6.109}$$

It follows from (6.109) the *Clausius equality*

$$\oint_{\Gamma} \left\{ \int_{\Omega_1} \frac{\delta q^e}{T} \, d\Omega \right\} d\Gamma = 0, \tag{6.110}$$

where Γ is a closed curve (cycle) in the space of the thermodynamical parameters.

For an irreversible process we obtain the inequality

$$\rho T \, ds \geq \delta q^e. \tag{6.111}$$

The difference

$$\delta q' = \rho T ds - \delta q^e \tag{6.112}$$

is called the *noncompensated heat.* It follows from the definition that this quantity is always non-negative. We now introduce the *density of dissipation* or *function of dissipation* $d \equiv w*$ by

$$\rho T ds = \delta q^e + w^* \, dt \tag{6.113}$$

(comp. with the definition (1.200)). It follows from this definition that $w^* \, dt = \delta q'$ and

$$w^* \geq 0. \tag{6.114}$$

We now obtain corollaries of the laws of thermodynamics useful for the solution of some problems in the modeling, e.g., for the control of correctness of the governing equations.

Let $\mathrm{i} = \mathrm{i}(x)$ be the power of the internal heat sources in a point x of the domain Ω_1, $\mathrm{i} = \mathrm{i}(x)$ is calculated per unit volume. The total increase of the heat inflow in the domain Ω_1 is the sum of the heat produced by the internal sources and inflow through the boundary $\Sigma_1 = \partial\Omega_1$, i.e.,

$$\delta Q^e = \left[\int_{\Omega_1} \mathrm{i} \, d\Omega - \int_{\Sigma_1} q \cdot \nu \, d\Sigma \right] dt, \tag{6.115}$$

where q is the vector of the *flux of heat* per unit of the surface area. The sign "minus" appears due to the fact that the vector ν is directed outside of the domain Ω_1.

Using the Gauss–Ostrogradski theorem in (6.115), we obtain the equation

$$\delta Q^e = \left[\int_{\Omega_1} (\mathrm{i} - \operatorname{div} q)\, d\Omega\right] dt = \int_{\Omega_1} \delta q^e \, d\Omega, \tag{6.116}$$

i.e.,

$$\delta q^e = (\mathrm{i} - \operatorname{div} q) dt. \tag{6.117}$$

Substituting the expression (6.117) in the equation of the heat inflow (6.107), we obtain:

$$\rho \frac{d\mathrm{u}}{dt} = \hat{\sigma} \cdot \cdot \hat{v} + \mathrm{i} - \operatorname{div} q + \frac{\delta q^*}{dt}. \tag{6.118}$$

The equation (6.118) can be used for the modeling of deformation of a solid. For example, the linear theory of thermoelasticity can be obtained with the following hypotheses:

1. $\delta q^* = 0$, $\delta q' = 0$
2. For the Fourier law for the vector of the flux of heat q it holds

$$q = -\hat{\kappa} \cdot \nabla T, \tag{6.119}$$

   where $\hat{\kappa}$ is the *tensor of the heat conductivity.* In an isotropic body $\hat{\kappa} = \kappa \hat{\delta}$
3. There exists a *natural state*, with $\hat{\varepsilon} = 0$, $\hat{\sigma} = 0$, $T = T_0$
4. All the thermodynamic functions depend on the parameters $(\hat{\varepsilon}, T')$ only, $T' = T - T_0$, and these dependencies can be approximate by a polynomial of degree two

Using these hypotheses, we rewrite the equation (6.107) in the form

$$d\mathrm{u} = \frac{1}{\rho} \hat{\sigma} \cdot \cdot d\hat{\varepsilon} + T ds + \frac{\mathrm{i}}{\rho}. \tag{6.120}$$

Let the function $\mathrm{f} = \mathrm{u} - sT$ be the *Helmholtz free energy.* Then it follows from (6.120) that

$$d\mathrm{f} = \frac{1}{\rho} \hat{\sigma} \cdot \cdot d\hat{\varepsilon} - s dT + \frac{\mathrm{i}}{\rho}. \tag{6.121}$$

With the notations

$$\rho \mathrm{u} = \bar{u}, \quad \rho \mathrm{f} = \psi, \tag{6.122}$$

we prove that the equations

$$\begin{aligned} \rho T &= \left(\frac{\partial \bar{u}}{\partial s}\right)_{\varepsilon}, & \rho s &= \left(\frac{\partial \psi}{\partial T}\right)_{\varepsilon}, \\ \hat{\sigma} &= \left(\frac{\partial \bar{u}}{\partial \hat{\varepsilon}}\right)_{s}, & \hat{\sigma} &= \left(\frac{\partial \psi}{\partial \hat{\varepsilon}}\right)_{T} \end{aligned} \tag{6.123}$$

hold. The lower indices means that the corresponding variable remains constant.

We now use the Taylor decomposition for the function ψ:

$$\begin{aligned}\psi(\hat{\varepsilon},T') =&\psi(0,T) + \frac{\partial\psi(0,T_0)}{\partial\hat{\varepsilon}}\cdot\cdot\hat{\varepsilon} + \frac{\partial\psi(0,T_0)}{\partial T'}T' + \left(\frac{\partial^2\psi(0,T_0)}{\partial\hat{\varepsilon}\partial\hat{\varepsilon}}\cdot\cdot\hat{\varepsilon}\right)\cdot\cdot\hat{\varepsilon}\\ &+ 2\left(\frac{\partial^2\psi(0,T_0)}{\partial\hat{\varepsilon}\partial T'}T'\right)\cdot\cdot\hat{\varepsilon} + \frac{\partial^2\psi(0,T_0)}{\partial(T')^2}(T')^2. \end{aligned} \tag{6.124}$$

It follows from the hypothesis on the natural state that

$$\frac{\partial\psi(0,T_0)}{\partial\hat{\varepsilon}} = 0, \quad \frac{\partial\psi(0,T_0)}{\partial T'} = 0. \tag{6.125}$$

Notate that

$${}^4\hat{a} = \frac{\partial^2\psi(0,T_0)}{\partial\hat{\varepsilon}\partial\hat{\varepsilon}} \tag{6.126}$$

(tensor of the elasticity modulus), and

$$\hat{\beta} = -\frac{\partial^2\psi(0,T_0)}{\partial\hat{\varepsilon}\partial T'} \tag{6.127}$$

(tensor of thermoelastic interaction).

It follows from the equations (6.123) and (6.124) that

$$\hat{\sigma} ={}^4 \hat{a}\cdot\cdot\hat{\varepsilon} - \hat{\beta}T', \tag{6.128}$$

the *Duhamel–Neimann law* for an anisotropic thermoelastic body.

For the isotropic body ($\hat{\beta} = \gamma\hat{\delta}$) we obtain

$$\hat{\sigma} = \lambda\nabla\cdot u\hat{\delta} + 2\mu\hat{\varepsilon} - \gamma\hat{\delta}T', \tag{6.129}$$

where λ and μ are the Lamé parameters.

Point out an experiment to evaluate the coefficient γ. Consider a stresses-free heating of an elastic body. It follows from the equation (6.129) and the condition $\hat{\sigma} = 0$ that

$$\hat{\varepsilon} = \frac{\gamma}{3K_V}\hat{\delta}T' = \alpha_T\hat{\delta}T' = \frac{\gamma_V}{3}\hat{\delta}T', \tag{6.130}$$

where γ_V is the *coefficient of the thermic volume extension*, α_T is the *coefficient of the linear thermic extension*, and K_V is the volume modulus of elasticity.

Suppose that all the thermoelastic characteristics of the body Ω are constant, and substitute the Duhamel–Neimann law (6.129) in the motion equation (6.72). We obtain

$$(\lambda+\mu)\nabla(\nabla\cdot u) + \mu\Delta u + \Phi = \rho\frac{\partial^2 u}{\partial t^2} + \gamma\nabla T'. \tag{6.131}$$

Note that the quantity $C_\varepsilon = \left(\frac{\partial\psi}{\partial T}\right)_\varepsilon$ is the *heat capacity* of a solid (see, e.g., [Now75]). Using the equations (6.123) and the Fourier law (6.119) in the equation of the heat conduction (6.107), we obtain

$$\varDelta T' - \frac{1}{\kappa_0}\frac{\partial T'}{\partial t} - r\frac{\partial}{\partial t}(\nabla u) = -\frac{\mathrm{i}}{\kappa\rho}, \tag{6.132}$$

where $\kappa_0 = \kappa/C_\varepsilon$, $r = \gamma T_0/\kappa$. (Recall that T_0 is the reference temperature and $T' = T - T_0$, T is the current value of the temperature.)

These equations represent the mathematical model of the interaction of two processes – mechanical deformation and heat conduction. It must be completed by the boundary and initial conditions (see, e.g., [Now75]). If we consider a contact problem then we obtain the unilateral constraints for the stresses and displacements at the contact boundary (see Chapter 4). So, we now can solve the contact problem with the heating of the contacting bodies due to the friction.

Unfortunately, a direct application of the above method to the contact problem is difficult, because all the transformations of the energy take place at the surface, not in the domain as it was supposed above. Appropriate method will be developed later in Section 6.4.

Substituting the expression (6.117) into the inequality (6.111) and using the Gauss formula, we obtain

$$\rho\frac{ds}{dt} \geq \frac{\mathrm{i}}{T}\operatorname{div}\left(\frac{q}{T}\right) = \frac{1}{T}(\mathrm{i} - \operatorname{div} q) + \frac{1}{T^2}q\cdot\nabla T. \tag{6.133}$$

Excluding the quantity $\mathrm{i} - \operatorname{div} q$ in the inequality (6.133) with (6.118), we obtain

$$\rho T\frac{ds}{dt} - \rho\frac{d\mathrm{u}}{dt} + \hat{t}\cdot\cdot\hat{v} - \frac{1}{T}q\cdot\nabla T + \frac{\delta q^*}{dt} \geq 0. \tag{6.134}$$

With the definition of the free energy $\psi = \mathrm{u} - Ts$ (see (6.122)) we can write (6.134) in the form

$$-\rho\frac{d\psi}{dt} + \hat{t}\cdot\cdot\hat{v} - \rho s\frac{dT}{dt} - \frac{1}{T}q\cdot\nabla T + \frac{\delta q^*}{dt} \geq 0. \tag{6.135}$$

It can be seen that the quantity on the left-hand side of the inequalities (6.134) and (6.135) is the dissipation w^*.

6.3.2 Extremum principles for dissipation and entropy

Equation of the increment of entropy

We now investigate the structure of the dissipation for some physical processes and show that in some models of continuum the dissipative function is a bilinear form (or functional). For this we return to the equation (6.118) written in the form:

$$\rho d\mathrm{u} = \hat{\sigma}\cdot\cdot d\hat{\varepsilon} + (\mathrm{i} - \operatorname{div} q)dt + \delta q^*. \tag{6.136}$$

Using the definition of entropy, we also find

$$\rho d\mathrm{u} = \hat{\sigma} \cdot \cdot d\hat{\varepsilon} + \rho T ds - \delta q' + \delta q^*. \tag{6.137}$$

Comparing the right-hand parts of the equations (6.136) and (6.137), we obtain the *equation of the increment of entropy*:

$$\rho T \frac{ds}{dt} = \mathrm{i} - \operatorname{div} q + w^* \tag{6.138}$$

Examples

Example 6.7. Consider a process of heat transfer in the nondeformed continuum, i.e., $d\varepsilon = 0$. Suppose that $w^* = 0$. Recall that the function w^* define the heat inflow due to transformation of the energy which differ from the heat inflow and mechanical work. For such a process the increment of entropy is

$$dS = \int_{\Omega_1} \rho ds = \left(\int_{\Omega_1} \frac{\mathrm{i}}{T} \, d\Omega - \int_{\Omega_1} \frac{\operatorname{div} q}{T} \, d\Omega \right) dt. \tag{6.139}$$

If $\mathrm{i} = 0$ then the equation (6.139) defines the increment of entropy due to the heat conductivity only. For such case we obtain the equation

$$\int_{\Omega_1} \rho \, ds = - \left(\int_{\Omega_1} \frac{\operatorname{div} q}{T} \, d\Omega \right) dt = - \left(\int_{\Sigma_1} \frac{1}{T} q \cdot \nu \, d\Sigma \right) dt - \left(\int_{\Omega_1} \frac{q \cdot \nabla T}{T^2} \, d\Omega \right) dt. \tag{6.140}$$

It follows from this equation that for an adiabatically isolated system the density of the increment of entropy is

$$\rho \, ds = - \frac{q \cdot \nabla T}{T^2}. \tag{6.141}$$

By definition, this quantity is the *heat dissipation*. Note that it is the bilinear form equal to the scalar product of the heat flow q and vector

$$\varPhi_q = - \frac{\nabla T}{T^2}, \tag{6.142}$$

which is called the *thermodynamic force* adjoint (conjugate) to the *flow* q.

We recall that we separate the state parameter π_0 (see Section 1.5) to investigate the heat transfer phenomena in detail. It can be seen from the previous results that the quantity (6.141) is the addition to the dissipation w^* introduced by the equation (6.113). This reasoning justify the definition of the heat dissipation.

Example 6.8. In a linearly viscous liquid governed by the generalized Newton equation

$$t_E^{ij} = -p\delta_{ij} + \varLambda(\nabla \cdot v)\delta_{ij} + 2M v_{ij} \equiv -p\delta_{ij} - P_{ij}^v, \tag{6.143}$$

where t_E^{ij} are the Euler components and Λ and M are the viscosity coefficients. (M is the coefficient of the shear viscosity and $\Lambda + 2/3M$ is the coefficient of the volume viscosity.) The dissipative function w^* is the bilinear form

$$w^* = q \cdot \Phi_q + p^v \Phi_v + \hat{P}^{Dv} \cdot \cdot \hat{\Phi}^{Dv}, \tag{6.144}$$

where p^v is the pressure arising due to the viscosity ($p^v = +1/3P^v_{kk}$), $\Phi_v = -\nabla \cdot v = -\operatorname{div} v$, $\hat{\Phi}^{Dv}$ is the deviator part of the deformations rate tensor $\hat{v}$.

The scalar p^v and tensor $\hat{P}^{Dv}$ are called the flows. The quantities Φ_v, $\hat{\Phi}^{Dv}$ are the forces conjugate to flows p^v and tensor $\hat{P}^{Dv}$. Hence, the dissipation w^* is the sum of products of flows and forces, i.e., is the bilinear form.

Note that in the work [PG05] it is demonstrated that in the linear theory of viscoelasticity (6.92) the dissipation is bilinear form, too, with the replacement of the number multiplication by a convolution in time.

Principle of minimum dissipation

We now generalize the previous results to a continuum, where there are f forces Φ_k and f corresponding flows J_k. Such a generalization is useful in applications for two reasons. First, if we have a mathematical model of a physical phenomena, we calculate the corresponding dissipation and verify the nonnegativity of the dissipative function w^*. If the nonnegativity requirement is violated, then the mathematical model is not correct.

Secondly, we can construct the mathematical model by definition of the dependencies of the flows on the forces or vice versa, from analysis of a experimental data and satisfying the nonnegativity requirement for the dissipation. We will use this way to construct the theory of contact with adhesion in the following section.

There exist extremal principles, the Onsager principle of minimum of energy dissipation, the Prigogine principle of minimum of entropy production, the Ziegler principle of maximum rate entropy production [Gya70], related to each other. These principles are also used for the mathematical modeling in the form of normality laws, see later.

We now suppose that the dissipation in a continuum is defined by the bilinear form

$$w^* = \sum_{k=1}^{f} J_k \Phi_k. \tag{6.145}$$

We emphasize that, despite the absence of additional notation, the values J_k, Φ_k can be scalars, vectors or tensors and the product $J_k \Phi_k$ can be given by the operation of number multiplication, scalar multiplication, double convolution or convolution in time of number parameters, vectors or tensors.

The construction of a mathematical model reduces to find the forces as the functions of flows (or vice versa). These dependencies are the *governing equations* [Sed97] or *complementary laws* [Ger73] . We consider first the linear dependencies of the forces in (6.145) on the flows (or inverse linear equations), and postulate that the dissipation w^* defined by (6.145) is nonnegative, and

$$J_i = \sum_{k=1}^{f} L_{ik}\Phi_k, \quad i = 1, \dots, f. \tag{6.146}$$

When the matrix of coefficients L_{ik} is diagonal, the physical processes described by the pairs of values $\{J_i, \Phi_i\}$ are independent of each other. These processes are connected due to the fact that they all contribute to the dissipative function (6.145). If $L_{ik} \neq 0$ for $i \neq k$ then the processes interact with each other.

Substituting the flows (6.146) in the expression for the dissipation (6.145), we obtain

$$w^* = \sum_{i=1}^{f}\sum_{j=1}^{f} L_{ij}\Phi_i\Phi_j \geq 0. \tag{6.147}$$

L. Onsager formulated, at the base of analysis of experimental data, the hypothesis on the symmetry of coefficient L_{ij}

$$L_{ij} = L_{ji} \quad \forall i, j, \tag{6.148}$$

which is called the *Onsager reciprocity principle.*

Using the theorem on the connection of the operator equation with the problem of functional minimization (see Sections 3.3 and 3.4), we conclude that the equation (6.146) is equivalent to the minimization of the function

$$\varphi(\Phi_1, \dots, \Phi_f) = \frac{1}{2} w^*(\Phi_1, \dots, \Phi_f) - \sum_{i=1}^{f} J_i\Phi_i. \tag{6.149}$$

This statement is called the *principle of minimum dissipation.* The flows J_i in the expression (6.149) are fixed and not varying in the process of minimization. We assume that the equation (6.146) can be solved with respect to the flows

$$\Phi_i = \sum_{k=1}^{f} R_{ik}J_k, \quad i = 1, \dots, f. \tag{6.150}$$

Then the expression for the dissipative function takes the form

$$w^* = \sum_{i=1}^{f}\sum_{j=1}^{f} R_{ij}J_iJ_j \geq 0. \tag{6.151}$$

The same theorems in Sections 3.3 and 3.4 lead, using the symmetry equations (6.148) and the inequality (6.151), to the conclusion about the equivalence of the relations (6.150) to the problem of the function minimization

$$\varphi^*(J_1, \dots, J_f) = \frac{1}{2} w^*(J_1, \dots, J_f) - \sum_{i=1}^{f} J_i \Phi_i, \tag{6.152}$$

as, opposed to (6.149), the forces Φ_i are fixed and flows are varied. The function w^* depends on the flows only (see the formula (6.151)).

This statement is the principle of minimum dissipation, too, but expressed with the flows. The formulations are local, as they characterize the process at a point of the domain Ω. Integration over this domain allows us state integral forms of the principle:

$$\int_{\Omega} \left[\frac{1}{2} w^*(\Phi_1, \dots, \Phi_f) - \sum_{i=1}^{f} J_i \Phi_i \right] d\Omega \longrightarrow \min_{\Phi}, \tag{6.153}$$

$$\int_{\Omega} \left[\frac{1}{2} w^*(J_1, \dots, J_f) - \sum_{i=1}^{f} J_i \Phi_i \right] d\Omega \longrightarrow \min_{J}. \tag{6.154}$$

In many applications, the governing equation relating the forces and flows are nonlinear, i.e.,

$$\{J_1, \dots, J_f\} = A(\{\Phi_1, \dots, \Phi_f\}), \tag{6.155}$$

where $A(\{\Phi_1, \dots, \Phi_f\})$ is an operator. If this operator satisfies the potentiality conditions, then we can formulate the governing equation (6.155) as the following variational principle:

$$\varphi(\Phi_1, \dots, \Phi_f) = \int_0^1 \sum_{i=1}^{f} A_i(\{t\Phi_1, \dots, t\Phi_f\}) \Phi_i \, dt - \sum_{i=1}^{f} J_i \Phi_i \longrightarrow \min_{\Phi} \tag{6.156}$$

with the nonvaried flows J_i. We can also find the analogous principle for the flows with the given forces Φ_i, using the relation inverse to (6.155).

Note, in conclusion, that these variational principles are related with the *principle of maximum of the rate of entropy* and with the *principle of the minimum of increment of entropy*, see, e.g., [Gya70].

6.3.3 Extremum principles in the theory of plastic flow

We now apply the approach of the previous section to a nonlinear continuum – to the theory of the elastic–plastic flow. We find that the governing equation has the form of the *normality law*.

We assume that the deformations are small and that the total tensor of the Cauchy deformations $\hat{\varepsilon}$ can be written as the sum of the elastic components $\hat{\varepsilon}^e$ and plastic components $\hat{\varepsilon}^p$:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p = A_{ijkl} \sigma_{kl} + \varepsilon_{ij}^p, \tag{6.157}$$

where A_{ijkl} are the components of the compliance tensor.

The main tool for the construction of governing equation is the surface loading – the geometric position of the points in stress space, separating the domain of pure elastic deformation from the domain where the plastic deformations can occur. The equation for this surface is set in the form

$$f(\sigma_{ij}, \varepsilon_{ij}^p, \chi_k^p) = 0, \tag{6.158}$$

where χ_k^p is a set of parameters, depending on the history of changes of the plastic deformations, which are fixed if ε_{ij}^p are fixed.

Negative values of the load function f correspond to the elastic (or rigid, it depends on the model) domain. We assume that the load surface is smooth, i.e., the function f is continuously differentiable. In unloading, when $d\varepsilon_{ij}^p = 0$, $d\chi_k^p = 0$, the accepted hypothesis leads to a constraint for the increments of stresses which is in the form of the strict inequality (due to D. Drucker, see, e.g., [Hil98]):

$$\frac{\partial f}{\partial \sigma_{ij}} \delta\sigma_{ij} < 0. \tag{6.159}$$

For a neutral loading f stays constant for small stress increments:

$$\frac{\partial f}{\partial \sigma_{ij}} \delta\sigma_{ij} = 0. \tag{6.160}$$

For an active loading, when $\delta\varepsilon_{ij}^p \neq 0$,

$$\frac{\partial f}{\partial \sigma_{ij}} \delta\sigma_{ij} > 0 \tag{6.161}$$

and, simultaneously,

$$\frac{\partial f}{\partial \sigma_{ij}} \delta\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}^p} \delta\varepsilon_{ij}^p + \frac{\partial f}{\partial \chi_k^p} \delta\chi_k^p = 0. \tag{6.162}$$

To find the connections between stresses and deformations, we use the principle of maximum of rate of change of mechanical work dissipation (see the previous paragraph):

$$(\sigma_{ij} - \tilde{\sigma}_{ij})\dot{\varepsilon}_{ij}^p \geq 0, \quad \forall \tilde{\sigma}_{ij}, \tag{6.163}$$

where an admissible stress $\tilde{\sigma}_{ij}$ satisfies the condition

$$f(\tilde{\sigma}_{ij}, \varepsilon_{ij}^p, \chi_k^p) \leq 0. \tag{6.164}$$

In the inequality (6.163), the rate of change of plastic deformations $\dot{\varepsilon}_{ij}^p$ is fixed, σ_{ij} stands for the true values of the stresses corresponding to the velocities $\dot{\varepsilon}_{ij}$. In the theory of plasticity, this principle is known as the *von Mises maximum principle.* The stress space is provided with the Euclidean metric with the scalar product

$$\langle \hat{\sigma}^{(1)}, \hat{\sigma}^{(2)} \rangle = \sigma_{ij}^{(1)} \sigma_{ij}^{(2)}.$$

The inequality (6.163) states that the angle between the vectors $\sigma_{ij} - \tilde{\sigma}_{ij}$ and $\dot{\varepsilon}^p_{ij}$ does not exceed $\pi/2$, thus implying the convexity of the surface f at the point of load. As $\sigma_{ij} - \tilde{\sigma}_{ij}$ can take any value, the vector $\dot{\varepsilon}^p_{ij}$ must be orthogonal to the vector $\tilde{\sigma}_{ij}$ at the point σ_{ij}, i.e., it must be orthogonal to the load surface. As the normal to the surface (6.167) is proportional to the gradient $\partial f/\partial\sigma_{ij}$, then we obtain the *normality law* (see the previous section):

$$\delta\varepsilon^p_{ij} = \delta\lambda\frac{\partial f}{\partial\sigma_{ij}}, \quad \delta\varepsilon^p_{ij} = \dot{\varepsilon}^p_{ij}dt. \tag{6.165}$$

This is the required connection between the "forces" and "flows," it is nonlinear and

$$\delta\lambda = -\left(\frac{\partial f}{\partial\sigma_{ij}}\delta\sigma_{ij} + \frac{\partial f}{\partial\chi^p_k}\delta\chi^p_k\right)\left(\frac{\partial f}{\partial\varepsilon_{mn}}\frac{\partial f}{\partial\sigma_{mn}}\right)^{-1}. \tag{6.166}$$

We notice that, in the 6D space of symmetric stress tensors, the relation (6.165) for $i \neq j$ is as follows:

$$2\delta\varepsilon^p_{ij} = \delta\lambda\frac{\partial f}{\partial\sigma_{ij}}.$$

In the so-called *perfect plasticity*, the function f depends on stresses only. In this theory, for the coefficient of the proportionality $\delta\lambda$, we have the following constraints: for an unload or neutral load $\delta\lambda = 0$ and for an active load $\delta\lambda \geq 0$.

Dual formulation

This allows us to determine the increments of stresses from the increments of deformations. The fundamental step of this formulation is to consider the surface of deformation F which constrains the domain of elastic deformations by means of the following inequality:

$$F(\varepsilon_{ij}, \varepsilon^p_{ij}, \chi^p_k) < 0. \tag{6.167}$$

We postulate the inequality

$$(\dot{\varepsilon}^p_{ij} - \tilde{\dot{\varepsilon}}^p_{ij})\varepsilon_{ij} \geq 0, \tag{6.168}$$

where $\tilde{\dot{\varepsilon}}^p_{ij}$ refers to any admissible kinematic value of the plastic deformations' change rates. The values of the total deformations ε_{ij} are fixed and the rates $\dot{\varepsilon}^p_{ij}$ correspond to the given values of deformations ε_{ij}. The definition of kinematic admissibility contains the following requirement: the vector $\tilde{\dot{\varepsilon}}^p_{ij}$ must be outward with respect to the surface $F = 0$, and if the deformation surface is smooth then the scalar product of $\tilde{\dot{\varepsilon}}^p_{ij}$ and the gradient F with respect to the variables ε_{ij} must be nonnegative.

Using the same reasonings which led to the relation (6.165), we can conclude that

$$\delta\varepsilon_{ij}^p = \delta\lambda \frac{\partial F}{\partial \varepsilon_{ij}}, \tag{6.169}$$

and that the function F is convex.

The equation (6.165) is called the *law of plastic flow associated with the surface* $f = 0$, and the equation (6.169) is called the *law of plastic flow associated with the surface of deformations* $F = 0$. The both are the form of the normality law.

We note that the principle (6.168) does not appear as a physical law, but the dual formulation allows us to relate it with the von Mises maximum principle.

To obtain a dual formulation, we use the following hypothesis: the dissipative function w^* depends on the accumulated plastic deformation ε_{ij}^p and on the rate $\dot{\varepsilon}_{ij}^p$ only, i.e.,

$$w^* = \sigma_{ij}\dot{\varepsilon}_{ij}^p = w^*(\dot{\varepsilon}_{ij}^p, \varepsilon_{ij}^p, \chi_k^p). \tag{6.170}$$

We consider the class of plasticity theories in which the governing equations do not depend on the choice of the time scale (in other words, they do not depend on the deformations' rates). For this class of materials, the dissipative function d must be homogeneous of first degree in the values $\dot{\varepsilon}_{ij}^p$. By the theorem on homogeneous functions:

$$w^* = \frac{\partial w^*}{\partial \dot{\varepsilon}_{ij}^p} \dot{\varepsilon}_{ij}^p. \tag{6.171}$$

Now we postulate the inequality

$$(\dot{\varepsilon}_{ij}^p - \tilde{\dot{\varepsilon}}_{ij}^p)\sigma_{ij} \geq 0, \tag{6.172}$$

which is the dual formulation of the von Mises maximum principle (6.163). In the inequality (6.172), the values of stresses σ_{ij} are fixed, the kinematically admissible plastic deformations rates $\dot{\varepsilon}_{ij}^p$ in the space of plastic deformations rates must lie inside the level surface of the function w^* corresponding to the true deformation rates, i.e.,

$$w^*(\tilde{\dot{\varepsilon}}_{ij}^p, \varepsilon_{ij}^p, \chi_k^p) \leq w^*(\dot{\varepsilon}_{ij}^p, \varepsilon_{ij}^p, \chi_k^p). \tag{6.173}$$

The inequality (6.172) leads to a statement about the convexity of the level surface of the function w^* and about the orthogonality of the vector σ_{ij} to the gradient of w^* with respect to the deformation rates:

$$\sigma_{ij} = \gamma \frac{\partial w^*}{\partial \dot{\varepsilon}_{ij}^p}. \tag{6.174}$$

The inequality (6.171) implies that $\gamma = 1$. In this case the formulations (6.169) and (6.174) are totally equivalent to each other.

Instead of the principle of maximum dissipation in the form of (6.168) or (6.172), D. Drucker and A. A. Ilyushin [Hil98] introduced the constraints in the form of inequalities on the contour integrals in the corresponding spaces.

The *Drucker postulate* takes the following form:

$$\oint_{\sigma} \Delta\hat{\sigma} \cdot \cdot d\hat{\varepsilon} \geq 0, \tag{6.175}$$

where $\Delta\sigma_{ij} = \sigma_{ij} - \sigma_{ij}^0$, σ_{ij}^0 is an arbitrary point in the stress space inside the load surface. It is assumed that integration in (6.175) is carried out along the closed curve in stress space (according to the assumption of A. A. Ilyushin in the deformation space).

Let us consider an infinitesimal cycle which is arranged partly outside of the stress surface. Using the formulated principle that the work done by the elastic deformations in a closed path is zero, we state that

$$(\sigma_{ij} - \sigma_{ij}^0)\delta\varepsilon_{ij}^p \geq 0. \tag{6.176}$$

After that, through the same methods as we used to get the law (6.169), we obtain the associated law of plastic flow.

6.3.4 Theory of normal dissipative mechanisms

First note that in the previous section we used a hypothesis on the smoothness of the load (or deformation) surface. This supposition is not necessary in the theory of *normal dissipative mechanisms* which can be found, e.g., in the textbook [Ger73]. We give here a brief exposition of this theory related with the main subject of our book.

We appeal to the fundamental inequality (6.134) (or (6.135)) for dissipative processes, and assume that the equilibrium state of a continuous medium is determined by $(m+1)$ parameters $\pi_0, \pi_1, \ldots, \pi_m$, where $\pi_0 = s$ is the entropy density. Particularly, for the interne energy u:

$$u = u(s, \pi_1, \ldots, \pi_m), \quad du = Tds + \sum_{p=1}^{m} \eta_p d\pi_p, \tag{6.177}$$

and

$$T = \frac{\partial u}{\partial s}, \quad \eta_p = \frac{\partial u}{\partial \pi_p}, \quad p = 1, \ldots, m. \tag{6.178}$$

We now introduce the so-called *axiom of the local state* [Ger73]:

Axiom of the local state.
The thermodynamic state of a neighborhood of some point in a continuum is completely determined by the specification of the internal energy u, temperature T, entropy s and other thermodynamic functions depending on the thermodynamical parameters $(\pi_0, \pi_1, \ldots, \pi_m)$ being the same as in the corresponding equilibrium state.

Substituting the expression (6.177) into the fundamental inequality (6.134), we obtain ($\delta q^* = 0$, see the definition (6.106))

$$\hat{t} \cdot \cdot \hat{v} - \rho \sum_{p=1}^{m} \eta_p \frac{d\pi_p}{dt} - \frac{1}{T} q \cdot \nabla T \geq 0. \tag{6.179}$$

So, the increase of dissipation is the bilinear form

$$\delta d = -\rho \sum_{p=1}^{m} \eta_p d\pi_p. \tag{6.180}$$

The equations

$$\eta_p = \eta_p(T, \pi_1, \ldots, \pi_m) \tag{6.181}$$

are called the *laws of a continuous medium state*. As was shown, the quantity on the left-hand side of the inequality (6.179) is the sum of the products of the forces $\eta_p \equiv J_k$ by the flows $d\pi_p/dt \equiv \Phi_k$ (see the expression (6.145)).

The equations which relate forces and flows are called *complementary laws* [Ger73] (see Section 6.3.2). The linear equations (6.146) and, particularly, the Fourier law of heat conduction are examples of the complementary laws. In this section we consider how to obtain additional laws in a pseudopotential form – in the form of (6.174) (or (6.165), (6.169)).

We use the notation d ($= w^*$) for the dissipative function. (The index $*$ means here the duality transformation.) Thus, let the expression

$$d = \sum_{k=1}^{f} J_k \Phi_k \tag{6.182}$$

be a continuous non-negative function of the variables $J_1, \ldots, J_f$, T, $\pi_1, \ldots, \pi_m$. This function depends on the variables $\pi_0, \pi_1, \ldots, \pi_m$. We now investigate the dependence of d on the variables $J_1, \ldots, J_f$. For brevity we introduce a vector $J = \{J_1, \ldots, J_f\}$ and the corresponding Euclidean metrics

$$J^{(1)} \cdot J^{(2)} = \sum_{k=1}^{f} J_k^{(1)} J_k^{(2)}. \tag{6.183}$$

We recall that these can be scalars, vectors and the second order tensors among the variable J_k. Multiplication in (6.182) and (6.183) can be ordinary, scalar or double convolution.

We consider the general case where $d = D(J)$, i.e., we postulate that there exists a function $d = D(J)$ which satisfies the following hypotheses:

1. The function $D(J)$ is convex, i.e.,

$$D(\alpha_1 J^{(1)} + \alpha_2 J^{(2)}) \leq \alpha_1 D(J^{(1)}) + \alpha_2 D(J^{(2)}) \tag{6.184}$$

   for any vectors $J^{(1)}$, $J^{(2)}$, and for any two positive numbers α_1, α_2, satisfying the condition $\alpha_1 + \alpha_2 = 1$.

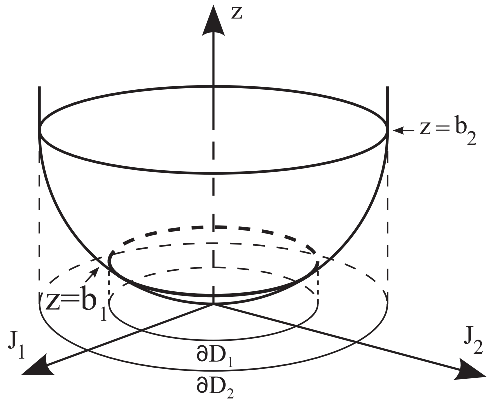

Fig. 6.1. Illustration of the quasi-homogeneity

2. The function $D(J)$ is quasi-homogeneous. To illustrate the property of quasi-homogeneity, we consider the graph of a function $z = D(J_1, J_2)$, domain $\mathcal{D}$ in the space $(\pi_1, \pi_2, \ldots, \pi_m, z)$ defined by

$$z \geq D(J), \tag{6.185}$$

and projections ∂D_1, ∂D_2 of the plane sections

$$D(J_1, J_2) = b_1, \quad D(J_1, J_2) = b_2, \quad b_1 = \text{const}, \; b_2 = \text{const} \tag{6.186}$$

onto the plan $z = 0$ (see Figure 6.1).
There exists a coordinate system and the number $\lambda \in \mathbb{R}^1$ such that, for any point $x_2 \in \partial D_2$, we can find only one point $x_1 \in \partial D_1$ such that

$$x_2 = \lambda x_1. \tag{6.187}$$

The curves ∂D_1, ∂D_2 with this property are called *homothetic*, as well the domains D_1, D_2 inside these curves. It is also supposed that these domains are convex and $D_1 \subset D_2$ if $b_2 > b_1$.

Let $h(J)$ be a positive homogeneous function of the first degree, i.e., $h(\lambda J) = \lambda h(J)$ for any positive number λ. To construct a normality law, we suppose that $h(J) = 1$ on the curve ∂D_1 and $h(J) = \lambda$ on ∂D_2, with an arbitrary choice of $b_2 \equiv b$. Then for an arbitrary J the function $d(J)$ can be defined as:

$$\begin{gathered} D(J) = b(\lambda), \quad \lambda = h(J), \\ \lambda > 0, \quad b(0) = 0, \quad b(1) = 1, \end{gathered} \tag{6.188}$$

where $b(\lambda)$ is a monotone increasing function of the variable λ, with $b(0) = 0$. It follows from the convexity of the domain $\mathcal{D}$ that the function $b(\lambda)$ is convex.

A dissipation phenomena is defined with the function of dissipation $d = D(J)$ being convex quasi-homogeneous, and by the *axiom of orthogonality*:

Axiom of orthogonality.
The force Φ corresponding to the flow J is orthogonal to the hyperplane of support at the boundary point J of the set $\partial\mathcal{D}$ (boundary of the domain $\mathcal{D}$), and is drawn out the domain $\mathcal{D}$.

We recall that the *hyperplane of support* is a hyperplane which contains at least one point of the set $\mathcal{D}$, and the whole set $\mathcal{D}$ lies totally in one of the closed half-spaces, given by this hyperplane. If the boundary $\partial\mathcal{D}$ of the set $\mathcal{D}$ is continuously differentiable, the hyperplane of support coincides with the tangent plane to the boundary.

Let ν be the outward drawn normal to the hyperplane of support. Then, the orthogonality hypothesis implies

$$\nu(J) \cdot (\tilde{J} - J) \geq 0, \quad \Phi = |\Phi|\nu \tag{6.189}$$

for all vectors $\tilde{J}$ satisfying

$$d(\tilde{J}) \leq b. \tag{6.190}$$

This is the *additional law* corresponding to the normal dissipative mechanism in the general case.

If the function $d(J)$ is continuously differentiable (as well as $b(\lambda)$ and $h(J)$), then the vector Φ is parallel to the gradient of the function d or h with respect to the variables J:

$$\Phi = \Lambda \operatorname{grad} h, \tag{6.191}$$

where Λ is the positive scalar coefficient of proportionality. In an angular point we have a set of the vectors $\{J\}$ situated between the extreme normals ν_1 and ν_2 (see Figure 6.2).

We prove that the factor Λ depends on the value $b(\lambda)$ only. For this purpose, we take the inner product of the both sides of (6.191) by J, using the formula (6.182) and the fact that h is the homogeneous function of the first degree:

$$J \cdot \Phi = \Lambda J \cdot \operatorname{grad} h = \Lambda h(J) = \Lambda\lambda = d(J) = b(\lambda). \tag{6.192}$$

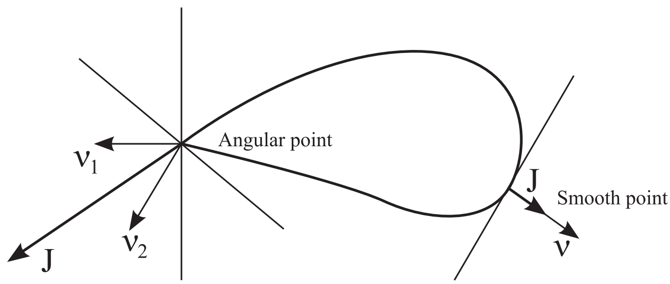

Fig. 6.2. Orthogonality law

Therefore,

$$\Lambda = \frac{b(\lambda)}{\lambda}, \quad \Phi = \frac{b(\lambda)}{\lambda} \operatorname{grad} h. \tag{6.193}$$

We introduce the function

$$\varphi(J) = \int_0^\lambda \frac{b(s)}{s} ds = \int_0^1 \frac{b(\lambda t)}{t} dt = \int_0^1 \frac{d(tJ)}{t} dt. \tag{6.194}$$

Equating (6.191) and (6.193), we get

$$\Phi = \operatorname{grad} \varphi. \tag{6.195}$$

The function φ is called the *pseudopotential of dissipation*. The prefix "pseudo" is added because the equality (6.195) is valid for active processes only.

Recalling the definition (5.100) of the subdifferential, we prove that, if the dissipative function and, hence, the pseudopotential φ also are nondifferentiable at some point J, the following inclusion occurs:

$$\Phi \in \partial\varphi(J). \tag{6.196}$$

Indeed, let the equation (6.196) hold. Then, according to the definition,

$$\varphi(\tilde{J}) - \varphi(J) \geq \Phi \cdot (\tilde{J} - J). \tag{6.197}$$

In the space of variables (z, J), we consider the domain Δ given by the inequality

$$z \geq \varphi(J) \tag{6.198}$$

and the surface

$$z = \varphi(J) + \Phi \cdot \tilde{J} - \Phi \cdot J \tag{6.199}$$

that appears to be plane to the set Δ at the point (z, J) due to the inequality (6.197). This surface crosses the z-axis at the point $\tilde{J} = 0$

$$z = z_0 = \varphi(J) - \Phi \cdot J. \tag{6.200}$$

On the other hand, after considering in the space (z, J) the section of the set Δ by the surface crossing the axis z and the vector J_0 which is defined above (the equality (6.188)), we conclude that the boundary of the set Δ in this hypersurface is given by the following equation:

$$z = \varphi(J) = \int_0^1 \frac{b(\lambda t)}{t} dt = a(\lambda), \tag{6.201}$$

where $a(\lambda)$ is a differentiable function, and the tangent to it crosses the z-axis at the point

$$z = z_0 = a(\lambda) - b(\lambda). \tag{6.202}$$

Comparing the expression (6.200) with (6.202) and taking into account that $\varphi(J) = a(\lambda)$, we conclude that

$$\Phi \cdot J = b(\lambda) = d(J), \tag{6.203}$$

i.e., indeed any vector Φ, satisfying the equation (6.196), corresponds to the vector J in the definition of a dissipative function. The converse is also true: any vector Φ satisfying the equation (6.203) belongs to the set of subgradients (i.e., the subdifferential) of the function φ (of the dissipation pseudopotential). In this reasoning, we can interchange the vectors Φ and J – there is duality between the force space and the flow space. The simplest way to obtain the fundamental equations is to apply the duality transformation (5.85), that was introduced before:

$$\varphi^*(\Phi) = \sup_J [J \cdot \Phi - \varphi(J)]. \tag{6.204}$$

The equality (5.113) and the definition (6.204) imply

$$J \in \partial\varphi^*(\Phi). \tag{6.205}$$

When the function $\varphi^*(\Phi)$ is continuously differentiable, the vector J is determined uniquely by the formula

$$J = \operatorname{grad} \varphi^*(\Phi). \tag{6.206}$$

The equations (6.205) and (6.206) are dual to the equations (6.196) and (6.195), stated earlier.

We obtain the geometrical interpretation of the inclusion (6.205) and its special case (6.206). For this purpose, we consider the set Δ of points (z, J), satisfying the inequality

$$z \geq \varphi(J).$$

Let us introduce some vector Φ and the plane surface of the set Δ at the point J, which is perpendicular to the vector $(-1, \Phi)$. We write its equation

$$z - \Phi \cdot J + \varphi^*(\Phi) = 0. \tag{6.207}$$

The points of contiguity of the plane surface and the set Δ (i.e., the vector J) are determined by the equation

$$\varphi(J) + \varphi^*(\Phi) = J \cdot \Phi. \tag{6.208}$$

This is the equation (5.113). The definition (6.204) follows from the assumption of the convexity of the function φ stated above.

To conclude this section, we note that if the cross effects are absent (this is the analogue of the diagonal matrices L_{ij} and R_{ij} in the relations (6.146) and (6.150)), normal dissipative mechanisms for various groups of variables lead to the additivity of the total dissipation, i.e., if $d^{(1)}(J)$ depends on some of the components of the vector J and $d^{(2)}(J)$ depends on the remaining components, then

$$d(J) = d^{(1)}(J) + d^{(2)}(J). \tag{6.209}$$

From the definitions, it is clear that

$$\varphi(J) = \varphi^{(1)} + \varphi^{(2)}, \quad \varphi^*(\Phi) = \varphi^{*(1)}(\Phi) + \varphi^{*(2)}(\Phi). \tag{6.210}$$

6.4 Variational theory of adhesion and crack initiation

The exhaustive treatment of the results of the crack motion process can be found in the monograph [Che79]. In this section we describe an approach which supplements some aspects of the existing research directions and points out new research methods in the mathematical analysis of the problems. We use the results obtained in [Fre82a, Fre82b, KI94, JKR71, BM82, Fre85, MB78].

6.4.1 Theory of adhesive joints formation and destruction

The classical formulation of contact problems, discussed above, does not take into account the adhesion forces, i.e., the contact pressure can be nonpositive only. In reality, the contact forces can be attractive. For example, we know the phenomena of adhesion of carefully polished metal plates, of metal cold welding in a vacuum, of the sticking of electrified surfaces. As an example of the last phenomenon we point out filmstrip sticking during rewinding, which was investigated by I. V. Obreimov (see [DKS73]).

For many metals and alloys, the Lennard–Jones potential is a satisfactory approximation of the surface interaction force potential. The Lennard–Jones potential leads to the following expression for the forces of central interaction:

$$F = \frac{12kT_1}{a}\left[\left(\frac{a}{r}\right)^7 - \left(\frac{a}{r}\right)^{13}\right], \tag{6.211}$$

where $k = 1,38 \cdot 10^{-16}\text{erg} \cdot K^{-1}$ is the Bolzmann constant, T_1 is the temperature (K), r is the current distance between atoms, and a is the equilibrium interval between the atoms. If $r < a$ then the force F is repulsive ($F \to -\infty$ if $r \to 0$). If $r > a$ then the force F is attractive. It follows from (6.211) that for $r = 2a$ the attractive force is about 1% of its maximum (Figure 6.3a).

Direct modeling of the contact interaction with the Lennard–Jones potential is a very difficult problem which can be solved, e.g., with a multiprocessors calculation technology (see, e.g., [RTHC98, BZX04]).

This is a reason to develop a simplified adhesion theory. In the scheme most used in applications, first proposed in [JKR71] (the so-called JKR-model or JKR-theory) and used in [Fre82a, Fre82b] and others, it is supposed that

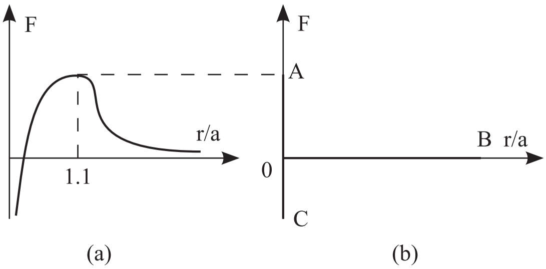

Fig. 6.3. Lennard–Jones potential and its schematization

the contact forces can be attractive but with a zero radius of action (see Figure 6.3b). In the classical model of contact, a point which represents the normal component of the contact force, moves only on the negative part of the axis F (along OC). If the normal component of the contact force can be positive (i.e., we use the JKR-model) then the corresponding point can move along the positive semiaxis OF up to the point A. The movement along the interval OA sometimes can be reversible, e.g., for contact interaction of polished surface. If we deal with fracture modeling, then this motion is irreversible from the point A (corresponding to maximal strength) to the point O (total fracture) only.

To describe the motion of a point representing the state of the contact force, along the interval OA, M. Fremond introduced in [Fre82a, Fre82b] a new state parameter β, corresponding to the adhesion force intensity. If $\beta = 1$ then the force of contact interaction can reach the point A (the adhesion bonds are not destroyed). If $\beta = 0$ then there is no adhesion, and the force can move only down from the point O.

Note that in the JKR-theory the force can move only backward and forward from O to A, i.e., the parameter β can be only 1 or 0. Thus, by definition

$$0 \leq \beta \leq 1. \tag{6.212}$$

Referring to the damage accumulation theory (see, e.g., [Rab69]), we can interpret the difference $1 - \beta$ as a scalar damage. This scalar damage satisfies an equation of the type

$$\frac{\partial \beta}{\partial t} = f(\beta, \hat{\sigma}, \ldots) \tag{6.213}$$

called the *damage kinetic equation*. This is an example of an additional law (see the definition in Section 6.3.4). The dots refer to the existence of additional arguments upon which the change of β depends.

To construct a closed theory, taking into account the damage accumulation (adhesion, in fact), we use the thermodynamics equations and methods described in Section 6.3.

6.4.2 Corollaries I and II of the laws of thermodynamics

Let us consider the most general formulation (6.102) of the first law assuming that

$$\frac{\partial E}{\partial t} = \frac{d}{dt}\left[\int_{\Omega} \rho \mathrm{u}\, d\Omega + \int_{\Omega\cap\Sigma_c} \mathrm{u}^c\, d\Sigma\right] + \int_{\Omega} \frac{1}{2}\rho v^2\, d\Omega, \tag{6.214}$$

where Ω is an arbitrary sub-domain of continuous medium in a domain Ω, Σ_c is the part of the surface which can contact the adhesive fixed rigid obstacle and u^c is the density of the internal energy on the segment of the boundary Σ_c. All other notations are as before. Introduce the quantities (comp. with the definitions (6.103) and (6.115))

- Mechanical work
$$\delta A^e = \int_{\Omega} \rho F \cdot \dot{u}\, dt\, d\Omega + \int_{\partial\Omega} (\hat{\sigma}\cdot\nu)\cdot \dot{u}\, dt\, d\Sigma \tag{6.215}$$
- Heat inflow
$$\delta Q^e = -\int_{\partial\Omega} q\cdot\nu\, dt\, d\Sigma + \int_{\Omega} \mathrm{i}\, dt\, d\Omega \tag{6.216}$$
- Energy inflow as a consequence of increase or decrease of the adhesive bonds
$$\delta Q^* = \int_{\partial\Omega\cap\Sigma_c} \mathrm{i}^c\, dt\, d\Omega \tag{6.217}$$

Substitute the expressions (6.214)–(6.217) in the equation (6.102), transform the second term in (6.207) according to the Gauss formula, and use the motion equation

$$\mathrm{div}\,\hat{\sigma} + \rho F = \rho\frac{d\dot{u}}{dt}. \tag{6.218}$$

As a result we obtain the equation

$$\begin{aligned}\frac{d}{dt}&\left[\int_{\Omega} \rho \mathrm{u}\, d\Omega + \int_{\partial\Omega\cap\Sigma_c} \mathrm{u}^c\, d\Sigma\right]\\ &= \int_{\Omega} \hat{\sigma}\cdot\cdot\hat{v}\, d\Omega + \int_{\Omega} \mathrm{i}\, d\Omega - \int_{\Omega} \mathrm{div}\, q\, d\Omega + \int_{\partial\Omega\cap\Sigma_c} \mathrm{i}^c\, d\Sigma.\end{aligned} \tag{6.219}$$

Using the arbitrariness in the choice of the subdomain Ω, we find from the equation (6.219) the heat inflow equation (see (6.107))

$$\rho\frac{du}{dt} = \hat{\sigma}\cdot\cdot\hat{v} + \mathrm{i} - \mathrm{div}\, q, \tag{6.220}$$

as well as a new equation – the equation of the surface energy balance at points of Σ_c

$$\frac{d\mathrm{u}^c}{dt} = \mathrm{i}^c. \tag{6.221}$$

Now we turn to the formulation (6.111) of the second law and define the total entropy S. We apply the following definition, using densities:

$$S = \int_{\Omega} \rho s \, d\Omega + \int_{\partial\Omega \cap \Sigma_c} s^c \, d\Sigma, \tag{6.222}$$

where the density s^c per unit area of the surface Σ_c is introduced together with the entropy density s per unit mass. Substituting the expressions (6.217) and (6.116) into the inequality (6.111), we get

$$\frac{d}{dt}\left[\int_{\Omega} \rho s \, d\Omega + \int_{\partial\Omega \cap \Sigma_c} s^c \, d\Sigma\right] \geq \int_{\Omega} \left[\frac{\mathrm{i}}{T} - \operatorname{div}\left(\frac{q}{T}\right)\right] d\Omega + \int_{\partial\Omega \cap \Sigma_c} \frac{\mathrm{i}^c}{T} \, d\Sigma. \tag{6.223}$$

Using the arbitrariness in choice of the form of the subdomain Ω, we obtain two local inequalities

$$\rho \frac{ds}{dt} \geq \frac{\mathrm{i}}{T} - \frac{1}{T} \operatorname{div} q - q \cdot \nabla\left(\frac{1}{T}\right), \quad \frac{ds^c}{dt} \geq \frac{\mathrm{i}^c}{T}. \tag{6.224}$$

If the values $\mathrm{i} - \operatorname{div} q$ and i^c are eliminated using the equations (6.220) and (6.221), we find

$$\rho T \frac{ds}{dt} \geq \rho \frac{d\mathrm{u}}{dt} - \hat{\sigma} \cdot\cdot \hat{v} - T q \cdot \nabla\left(\frac{1}{T}\right), \quad T \frac{ds^c}{dt} \geq \frac{d\mathrm{u}^c}{dt}. \tag{6.225}$$

The density ψ of free energy at points of the domain Ω and ψ^c at points of the surface Σ_c are given by

$$\psi = \mathrm{u} - T s, \quad \psi^c = \mathrm{u}^c - T s^c. \tag{6.226}$$

We eliminate the values u and u^c in the inequality (6.225) to find

$$\rho \frac{d\psi}{dt} \leq \hat{\sigma} \cdot\cdot \hat{v} + T q \cdot \nabla\left(\frac{1}{T}\right) - \rho s \frac{dT}{dt}, \quad \frac{d\psi^c}{dt} \leq -s^c \frac{dT}{dt}. \tag{6.227}$$

Integrating the first of these inequalities over the domain Ω and the second inequality over $\partial\Omega \cap \Sigma_c$ and summing the results, we get the *Clausius–Duhem inequality*:

$$\begin{aligned} \int_{\Omega} \rho \frac{d\Psi}{dt} \, d\Omega + \int_{\partial\Omega \cap \Sigma_c} \frac{d\Psi^c}{dt} \, d\Sigma \leq \int_{\Omega} \hat{\sigma} \cdot\cdot \hat{v} \, d\Omega \\ + \int_{\Omega} T q \cdot \nabla\left(\frac{1}{T}\right) d\Omega - \int_{\Omega} \rho s \frac{dT}{dt} \, d\Omega - \int_{\partial\Omega \cap \Sigma_c} s^c \frac{dT}{dt} \, d\Sigma. \end{aligned} \tag{6.228}$$

If it is possible to change the surface energy (e.g., through electrification) then at the right-hand side of the equation (6.221) an additional term appears. This term refers to the described change in the form $A\dot{\beta}$, where A is a proportionality coefficient depending on contact surface properties.

In this case we have an additional contribution in the part δQ^* of the energy inflow to the system, i.e.,

$$\delta Q^* = \int_{\partial\Omega\cap\Sigma_c} (\mathrm{i}^c + A\dot{\beta})\, dt\, d\Sigma, \tag{6.217'}$$

Suppose that the punch can move. The displacements of the punch $u = u_s$ are prescribed on the surface which can contact Σ_c. Denote the prescribed surface force by P.

To obtain the necessary corollaries of the laws of thermodynamics, we use the principle of virtual power [Ger73]. The virtual power $\wp_i$ of the internal forces in the system is

$$\wp_i = -\int_\Omega \hat{\sigma}\cdot\cdot\hat{\varepsilon}(\dot{v})\, d\Omega - \int_{\partial\Omega\cap\Sigma_c} [F_\beta\dot{\gamma} + Q\cdot(\dot{v} - \dot{v}_s)]\, d\Sigma, \tag{6.229}$$

where F_β is a parameter conjugate to the adhesion parameter β. If we consider β as a kinematic characteristic of the system, then F_β can be considered as a force parameter, because, by definition, the quantity $F_\beta\dot{\beta}$ is the power density. As usual, we distinguish between the actual and admissible state parameters: $\dot{v}$ is the kinematically admissible velocity determined as in Section 4.3.2 (see the formula (4.158)) with the constraint $\dot{u}_s = \dot{v}$ on Σ_c. We denote by $\dot{v}_s$ an admissible field $\dot{u}_s$, which corresponds to the displacement of the punch as a rigid system. Q is the density of the force acting on the punch by the body Ω, $\dot{\gamma}$ is an admissible change of the parameter $\dot{\beta}$. Constraints for this parameter follow from the definition (6.212): $\dot{\gamma} \geq 0$ when $\beta = 0$ (β can increase only), $\dot{\gamma} \leq 0$ when $\beta = 1$ (β can decrease only).

The virtual power of the external loads is

$$\wp_e = \int_\Omega (\rho F - \rho\ddot{u})\cdot\dot{v}\, d\Omega + \int_{\partial\Omega} P\cdot\dot{v}\, d\Sigma + \int_{\partial\Omega\cap\Sigma_c} [A\dot{\gamma} + g_S\cdot\dot{v}_S]\, d\Sigma. \tag{6.230}$$

It follows from the definitions that P is the prescribed force density at the points Σ_σ. On the rest of the surface the function P is unknown and g_S is the punch reaction.

Considering the equation of virtual power $\wp_e + \wp_i = 0$ and using the arbitrariness in the choice of the subdomain Ω and of the kinematic parameters $\dot{v}$, $\dot{v}_S$, $\dot{\gamma}$, we get the equations (6.218), the boundary condition on Σ_σ, and the relations

$$F_\beta = A, \quad \hat{\sigma}\cdot\nu = P - Q, \quad Q = -g_s, \tag{6.231}$$

which holds on Σ_c.

Repeating the reasonings which gave the equations (6.220) and (6.221), we state that the equation (6.220) does not change. The equation (6.221) takes the following form:

$$\frac{du^c}{dt} = \mathrm{i}^c + F\dot{\beta} + Q\cdot(\dot{u} - \dot{u}_S). \tag{6.221'}$$

In the Clausius–Duhem inequality (6.228) on the right, an additional term

$$\int_{\partial\Omega\cap\Sigma_c} [F_\beta\dot{\beta} + Q\cdot(\dot{u}-\dot{u}_S)]\, d\Sigma \tag{6.232}$$

appears. For isothermal processes ($T = \text{const}$) we have

$$\int_\Omega \rho\frac{d\Psi}{dt}\, d\Omega + \int_{\partial\Omega\cap\Sigma_c} \frac{d\Psi^c}{dt}\, d\Sigma \leq \int_\Omega \hat{\sigma}(u)\cdot\cdot\hat{\varepsilon}(\dot{u})\, d\Omega + \int_{\partial\Omega\cap\Sigma_c} [F_\beta\dot{\beta} + Q\cdot(\dot{u}-\dot{u}_S)]\, d\Sigma. \tag{6.233}$$

For simplicity here and later, consider the case of small disturbances of the displacements and their gradients.

Recall that we distinguish between the state equations and additional laws [Ger73]. Construction of the state equations, valid for all the materials with a given set of the state parameters, reduces to the determination of the free energy as the function of these parameters. Construction of the additional laws reduces to the determination of the dissipation (defined as the difference between the right- and left-hand parts of the inequality (6.233)) as a function (or functional) depending on the same parameters. We emphasize that a state equation always reflects a conservation law. We begin the construction of the mathematical model of adhesion contact with the state equation.

6.4.3 State equations

To construct the state equations, we will use the hypothesis about the local thermodynamic equilibrium [Ger73]. For simplicity, we use the hypothesis that the temperature is constant and

1. The total system of thermodynamic parameters of a state consists of the Cauchy tensor of the small deformations $\hat{\varepsilon}(u) = 1/2(\nabla u + \nabla u^T)$, the cohesion parameter β, and the punch displacement u_s
2. $\psi = \psi(\hat{\varepsilon})$, $\psi^c = \psi^c(\beta, u_s)$, which are convex functions of their arguments
3. The constraints
$$0 \leq \beta \leq 1, \quad \beta u_s = 0, \quad x \in \Sigma_c \tag{6.234}$$
4. The functional
$$\Psi = \int_\Omega \rho\psi\, d\Omega + \int_{\Sigma_c} \psi^c\, d\Sigma, \tag{6.235}$$
the free energy of the system computed by the formula (6.235) only at those points where the constraints (6.234) are satisfied (in addition, to the impenetrability condition on Σ_c).

Denote the set of parameters satisfying all these constraints by K, and extend the definition (6.235) to all the values of these parameters by supposing that $\Psi = +\infty$ outside the subset K (the price of such extension is the non-differentiability of Ψ) and the need to use the subgradient operator instead of the Gâteaux-derivative (see the next section). Note that the set K is nonconvex due to the constraints (6.234), the functional Ψ will not be convex either.

Let us introduce the thermodynamic forces $\tilde{\tau}$, G_1, G_2, which are adjoint to the thermodynamic parameters $\hat{\varepsilon}$, u_s, β. The assumptions 1–4 lead to the following state equations:

$$\{\hat{\tau}(x \in \Omega), G_1(x \in \Sigma_c), G_2(x \in \Sigma_c)\} \in \partial\Psi, \tag{6.236}$$

where $\partial\Psi$ is the subdifferential of Ψ. We recall the definition

$$\partial\Psi = \Big\{\{\hat{\tau}, G_1, G_2\} \mid \Psi(\tilde{\hat{\varepsilon}}, \tilde{u}_s, \tilde{\gamma}) \geq \Psi(\hat{\varepsilon}, u_s, \gamma) + \langle \hat{\tau}, \tilde{\hat{\varepsilon}} - \hat{\varepsilon}\rangle \\ + \langle G_1, \tilde{u}_s - u_s\rangle + \langle G_2, \tilde{\gamma} - \gamma\rangle \; \forall \{\tilde{\hat{\varepsilon}}, \tilde{u}_s, \tilde{\gamma}\} \in K\Big\}, \tag{6.237}$$

where

$$\langle \hat{\tau}, \hat{\varepsilon}\rangle = \int_\Omega \hat{\tau} \cdot\cdot \hat{\varepsilon}\, d\Omega, \quad \langle G_1, u_s\rangle = \int_{\Sigma_c} G_1 \cdot u_s\, d\Sigma, \quad \langle G_2, \gamma\rangle = \int_{\Sigma_c} G_2 \gamma\, d\Sigma.$$

If the function Ψ is differentiable then the definition (6.237) implies that

$$\hat{\tau} = \frac{1}{\rho}\frac{\partial\Psi}{\partial\hat{\varepsilon}}. \tag{6.238}$$

We emphasize that the analogous formulae for G_1, G_2 cannot be written, because the functional Ψ is nondifferentiable in the variables u_s, β.

The expression (6.238) implies that the tensor $\hat{\tau}$ has the dimensionality of stress, and – as it will be clear – if the dissipation is absent, $\hat{\tau}$ is equal to the total tensor of stresses $\hat{\sigma}$. The physical meaning of the forces G_1, G_2 will be explained later, see the examples later in this chapter.

Let t and $t+\Delta t$ be two close values of time variable, $\Delta t \to 0$. Choose in the inequality (6.237) $\hat{\varepsilon}$, u_s, γ for the value t, and $\tilde{\hat{\varepsilon}} = \hat{\varepsilon}(t+\Delta t)$, $\tilde{u}_s = u_s(t+\Delta t)$, $\tilde{\gamma} = \gamma(t+\Delta t)$. After dividing by Δt, calculating of limit with $\Delta t \to 0$, and repeating this calculation with $\Delta t \leftarrow -\Delta t$, we find the equation

$$\frac{d\Psi}{dt} = \int_\Omega \hat{\tau} \cdot\cdot \hat{\varepsilon}(\dot{u})\, d\Omega + \int_{\Sigma_c} (G_1 \cdot \dot{u}_s + G_2\dot{\beta})\, d\Sigma, \tag{6.239}$$

in which Ω is the whole domain.

Comparing the expressions (6.235) and (6.239) and the left-hand part of the Clausius–Duhem inequality (6.233), we obtain the inequality:

$$D = \int_\Omega (\hat{\sigma} - \hat{\tau}) \cdot\cdot \hat{\varepsilon}(\dot{u})\, d\Omega + \int_{\Sigma_c} [(Q - G_1)\cdot \dot{u}_s + (F_\beta - G_2)\dot{\beta}]\, d\Sigma \geq 0, \tag{6.240}$$

which will be used later.

The values $\hat{\tau}$, G_1, G_2, determined by the equations (6.236), are called the *reversible parts of the forces* $\hat{\sigma}$, Q, F_β. The differences in the expression for dissipation in (6.240) (e.g., $F_\beta - G_2$) are called the *irreversible parts of the forces*. This concept of decomposition force parameters into sums of invertible and non-invertible parts was introduced by H. Ziegler [Zie83]. Particularly, it is implemented in the theory of elastic-plastic material, using the hypothesis on the additivity of elastic and plastic deformations.

6.4.4 Johnson–Kendall–Roberts theory of adhesion (JKR theory)

This theory was published in [JKR71]. Some analytical solutions can be find in [Joh85]. A variational formulation for such problems was given by M. Fremond [Fre82a, Fre82b].

Consider the contact of a deformable body with a fixed rigid stamp. The boundary of the stamp is described by $\Psi(x) = 0$ (see the explication and hypotheses on the function Ψ in Section 4.2). The essential hypothesis of the JKR theory consists of the statement that in a contact point with $\Psi(x + u(x)) = 0$, $x \in \Sigma_c$, any point of the rigid stamp move together with the contact point of the deformed body. If the contact is violated then the contact force is zero. Therefore, at a contact point we have the following boundary conditions:

$$u(x) = u_S \tag{6.241}$$

and

$$\hat{\sigma} \cdot \nu \neq 0. \tag{6.242}$$

Outside of the contact zone the following equation holds:

$$\hat{\sigma} \cdot \nu = 0. \tag{6.243}$$

This condition means that there is no intermediary case corresponding to a partially destruction of adhesion bonds. In other words, the parameter β in the Fremond variational theory can be 1 or 0 only. The domain with nonzero adhesion force is determined by the boundary displacement continuity requirement.

Construction of the free energy and dissipation in JKR theory

We assume that

1. $u_s = 0$ (stamp is fixed)
2. The parameter β can be 1 or 0 only

We suppose that there is no dissipation, i.e., $D = 0$ (irreversible parts of all the forces are zero). Therefore,

$$\hat{\sigma} = \hat{\tau}, \quad Q = G_1, \quad F_\beta = G_2. \tag{6.244}$$

Theorem 6.9. *The JKR theory follows from all these hypotheses if we assume that the free energy is*

$$\Psi = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega} \hat{\varepsilon} \cdot \cdot {}^4 a \cdot \cdot \hat{\varepsilon}\, d\Omega - \int_{\Sigma_c} w\beta\, d\Sigma, & \text{if } \{u,\beta\} \in K, \\ +\infty, & \text{if } \{u,\beta\} \notin K, \end{cases} \tag{6.245}$$

where

$$K = \{\{v,\gamma\} \mid v = v(x),\ x \in \Omega;\ v|_{\Sigma_u} = 0;\ v \cdot \nu|_{\Sigma_c} \le 0; \\ \gamma = \gamma(x),\ x \in \Sigma_c;\ 0 \le \gamma \le 1,\ \gamma v = 0\}, \tag{6.246}$$

K is a set of the admissible values of parameter, and $w = \text{const} > 0$ *is the so-called* constant of Dupré adhesion *(see a definition, e.g., in [DKS73]).*

Proof. For simplicity we neglect the force of inertia and the parameter A. In the inequality (6.237) we make the replacements

$$\tilde{\hat{\varepsilon}} = \hat{\varepsilon}(v), \quad \hat{\varepsilon} = \hat{\varepsilon}(u), \quad \tilde{\gamma} = \gamma, \quad \gamma = \beta.$$

The inequality in the definition (6.237) now becomes

$$\frac{1}{2}\int_{\Omega} [\hat{\varepsilon}(v) \cdot \cdot {}^4\hat{a} \cdot \cdot \hat{\varepsilon}(v) - \hat{\varepsilon}(u) \cdot \cdot {}^4 a \cdot \cdot \hat{\varepsilon}(u)]\, d\Omega + \int_{\Sigma_c} [(-w)(\gamma - \beta)]\, d\Sigma \\ \ge \int_{\Omega} \hat{\sigma}(u) \cdot \cdot \hat{\varepsilon}(v-u)\, d\Omega. \tag{6.247}$$

The Gauss formula, the equations (6.218) and the condition on Σ_c lead to the inequality

$$\frac{1}{2}\int_{\Omega} [\hat{\varepsilon}(v) \cdot \cdot {}^4\hat{a} \cdot \cdot \hat{\varepsilon}(v) - \hat{\varepsilon}(u) \cdot \cdot {}^4 a \cdot \cdot \hat{\varepsilon}(u)]\, d\Omega - \int_{\Sigma_c} w(\gamma - \beta)\, d\Sigma \\ \ge \int_{\Omega} \rho F \cdot (v-u)\, d\Omega + \int_{\Sigma_c} P \cdot (v-u)\, d\Sigma \equiv L(v-u). \tag{6.248}$$

Introducing the functional

$$\Phi(\gamma, v) = \frac{1}{2}\int_{\Omega} \hat{\varepsilon}(v) \cdot \cdot {}^4 a \cdot \cdot \hat{\varepsilon}(v)\, d\Omega - \int_{\Sigma_c} w\gamma\, d\Sigma - L(v) \equiv J(v) - \int_{\Sigma_c} w\gamma\, d\Sigma, \tag{6.249}$$

from (6.248) we obtain the inequality

$$\Phi(\gamma, v) \ge \Phi(\beta, u) \quad \forall \{\gamma, v\} \in W(\beta, u),\ \{\gamma, v\} \in K\{\beta, u\}, \tag{6.250}$$

which means that $\{\beta, u\}$ is a local minimum of the functional Φ on the set K. $W(\beta, u)$ stands for the neighborhood of the point $\{\beta, u\}$, for which the definition (6.237) is valid. The subdifferential $\partial\Psi$ in (6.237) is local. The problem of computing a local minimum of the functional (6.249) can be simplified using the following statement: the parameter β can take only extreme values $\beta = 0$ or $\beta = 1$. To prove this, we consider a point $x \in \Sigma_c$ where contact occurs, i.e., $\Psi(x + u(x)) = 0$. We assume that the point x is surrounded by an area O of non-zero measure, where the contact occurs, too. In the inequality (6.250), we set $v = u$, $\gamma = \beta$ on the complement of the area O to the set Σ_c. Then

$$\int_O w(\beta - \gamma)\, d\Sigma \geq 0 \quad \forall\gamma,\ 0 \leq \gamma \leq 1. \tag{6.251}$$

The integral is nonnegative iff

$$\beta = 1. \tag{6.252}$$

At the points where we can have $\beta < 1$ values of γ in a neighborhood of the point β are arbitrary, and from the inequality (6.251) it follows that $\beta = 0$.

Suppose that $w =$ const, and denote by $S_c(v) \equiv S(\Sigma_c^c)$ the area of the part $\Sigma_c^c \subseteq \Sigma_c$, where $\beta = 1$. Then, it follows from the inequality (6.250) and the condition (6.252) on Σ_c^c that

$$J(v) - wS_c(v) \geq J(u) - wS_c(u) \quad \forall v \in K_1, \tag{6.253}$$

$$K_1 = \{v \mid v = v(x),\ x \in \Omega;\ v|_{\Sigma_u} = 0;\ v \cdot \nu|_{\Sigma_c} \leq 0\}. \tag{6.254}$$

The obtained statement of a problem in the adhesion theory is the *JKR theory* [JKR71]

Note that by the same calculation we obtain from the inequality (6.250) the following generalization of the JKR theory

$$J(v) - \int_{\Sigma_c^c} w\, d\Sigma \geq J(u) - \int_{\Sigma_c^c} w\, d\Sigma, \quad \forall v \in K_1, \tag{6.255}$$

which allows us to consider the problems with $w \neq$ const.

6.4.5 Examples in the JKR theory

Note that the main difficulty in the solution of an adhesion contact problem is that the minimum of the functional in (6.253) or in (6.255) is local. In other words, different cohesion zones may correspond to the same state of the external loads, i.e., the solution depends on the history of changes of these loads. If we construct a solution or numerical algorithm we must take this into account.

Example 6.10. Consider the simplest one-dimensional problem on the separation of a thin film from a fixed rigid foundation (see Fig. 6.4). Suppose that the film is inextensible, and denote its length by l. Consider a current state of the system characterized by an undestroyed part of the adhesive joint of length

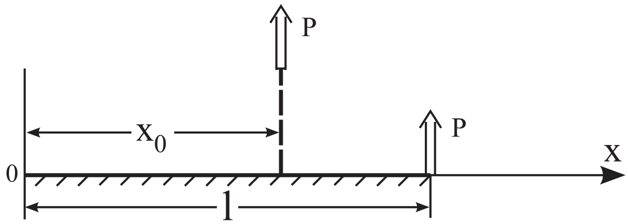

Fig. 6.4. The problem of the separation of a thin film

x_0. The work of the applied force P on the path from the initial state (where $x_0 = l$) to the current state is $P(l - x_0)$. Then the functional $J(v) - wS_c(v)$ in (6.253) is

$$J_w = J(x_0)_w = x_0(P - w) - Pl. \tag{6.256}$$

If we suppose that in (6.256) $P < w$, then the minimum of J corresponds to $x_0 = l$, i.e., there is no separation. If $P > w$ then $x_0 = 0$ and the film is completely released. The critical value is

$$P = P_* = w, \tag{6.257}$$

when x_0 can take any values depending on the prescribed displacement of the point of application of the force P.

Example 6.11. Consider the same problem as in the previous example. Suppose now that the film is an elastic film, i.e.,

$$P(x) = \tilde{E}\frac{du}{dx}, \quad \tilde{E} = ES, \tag{6.258}$$

where E is the Young modulus of the film, S is the area of the film cross section, $u = u(x)$ is the displacement, and du/dx is the deformation of the film.

By supposition, the foundation is rigid, $P(x) =$ const. Then the solution of the differential equation (6.258) with the boundary condition

$$u(x_0) = 0 \tag{6.259}$$

is

$$u(x) = \frac{P}{\tilde{E}}(l - x_0). \tag{6.260}$$

We now add the energy of elastic deformation, which corresponds to the solution (6.260), to the function (6.256), and reduce the problem to the minimization of the function

$$J_{w1} = J(u(l), x_0)_{w1} = \frac{\tilde{E}[u(l)]^2}{2(l - x_0)} wx_0 - P[l - x_0 + u(l)]. \tag{6.261}$$

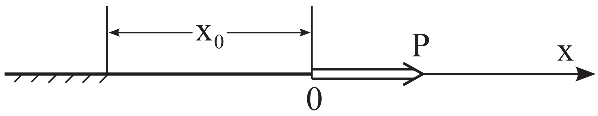

Fig. 6.5. Separation of a film from an elastic foundation

The critical value of the force P corresponding to $x_0 = 0$ is

$$P = P_* = -\tilde{E} + \sqrt{\tilde{E}^2 + 2\tilde{E}w}. \tag{6.262}$$

If $w \to 0$, $\tilde{E} \to +\infty$, and

$$\lim_{w\to 0,\, \tilde{E}\to+\infty} \tilde{E}w = \tilde{w},$$

where $\tilde{w}$ is as in the previous example, then, taking into account the approximation

$$\sqrt{\tilde{E}^2 + 2\tilde{E}w} \approx \tilde{E}w,$$

which is valid for the small values of w, we obtain the result (6.257) again.

Example 6.12. We now estimate the influence of the elasticity foundation. Consider a problem on the separation of a thin film from an elastic foundation by a force parallel to the film (see Figure 6.5). We choose the axis Ox as in Figure 6.5. Then the film occupies the domain $-\infty \le x \le 0$. Film is loaded by the force P, oriented along the axis Ox and applied to the point O.

We assume that the part $x_0 \le x \le 0$ is free. The Hooke law for the film and the Winkler hypothesis on the foundation leads to the following equations:

$$\begin{cases} \tilde{E}u'' - ku = 0, & -\infty < x < x_0, \\ \tilde{E}u'' = 0, & x_0 < x < 0 \end{cases} \tag{6.263}$$

with the boundary conditions

$$\begin{cases} \tilde{E}u' = P, \quad x = 0 \\ u(x_0 - 0) = u(x_0 + 0), \quad u'(x_0 - 0) = u'(x_0 + 0), \end{cases} \tag{6.264}$$

$$\lim_{x\to-\infty} u(x) = 0, \tag{6.265}$$

where $k = \text{const} > 0$ is the elasticity coefficient of the foundation.

The solution of the boundary value problems, (6.263) and (6.264) is

$$u(x) = \begin{cases} \dfrac{P}{\tilde{E}\lambda} e^{\lambda(x-x_0)}, & -\infty < x \le x_0 \\ \dfrac{P}{\tilde{E}}\left(x - x_0 + \dfrac{1}{\lambda}\right), & x_0 \le x \le 0, \ \lambda = \sqrt{k/\tilde{E}}. \end{cases} \tag{6.266}$$

From this solution we obtain the following formulae for the variations:

$$\delta\left[\frac{1}{2}\int_{-\infty}^{0}\tilde{E}(u')^2dx\right] = -\frac{P_0^2}{2\tilde{E}}\delta x_0,$$
$$\delta\left[\frac{1}{2}\int_{-\infty}^{x_0}ku^2dx\right] = 0, \quad \delta\left[\int_{\infty}^{x_0}w\beta dx\right] = w\delta x_0. \tag{6.267}$$

The corresponding critical value of the force

$$P_* = \sqrt{2\tilde{E}w} \tag{6.268}$$

does not depend on k.

Solution of more complicate contact problems on the stable or critical state of the heated coat on a elastic half-space was obtained, with the boundary element method (BEM), in [Iva89].

Adhesive contact of a disc and a half plane

We consider the plane problem on the indentation of a round disc with radius R into the elastic half plane $y \geq 0$. The analogous problem for the axially symmetrical problem was first investigated in [JKR71].

If the boundary of the cohesion zone is described by a set of parameters $p_1, \dots, p_m$, then a general solution algorithm for such a problem can be constructed as follows. Note first that, in the considered case, the energy functional $J(\Sigma_c^c)$ is a function of the parameters $p_1, \dots, p_m$. If the minimum is obtained at the internal point of the set $K(p)$, $p = \{p_1, \dots, p_m\}$, of the admissible values of the parameters $p_1, \dots, p_m$, then the problem reduces to the following system of algebraic equations:

$$\frac{\partial J}{\partial P_i} = 0, \quad i = 1, \dots, m. \tag{6.269}$$

If the minimum is obtained at the boundary point $p = p_0$ of the set $K(p)$, then we need to apply the appropriate algorithms of nonconvex optimization.

We will see that in the plane problem on the indentation of a round disc into the elastic half plane there is one parameter p only, and there are three local minima determined by the equation (6.269).

We use as the axis Ox the straight line which coincides with the boundary of the half plane. Let the point O be at the center of the cohesion zone and l be the halflength of the cohesion zone. For any value of l, the contact pressure distribution is (see, e.g., [Mus53])

$$\sigma_N = \sigma_N(x) = \frac{2\mu(l^2 - 2x^2)}{R(\kappa+1)\sqrt{l^2 - x^2}} + \frac{P_0}{\pi\sqrt{l^2 - x^2}}, \tag{6.270}$$

where μ is the shear modulus, ν is the Poisson coefficient, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 - \nu)$ for plane stress, and P_0 is the pressing force. In this problem the functional $J(\Sigma_c^c)$ is a function of the parameter l:

$$J(l) = \frac{\mu\pi l^4}{8R^2(\kappa + 1)} - \frac{P_0 l^2}{8R} - 2wl. \tag{6.271}$$

The minimization of the functional (6.271) with respect to the variable l leads to the cubic equation

$$\frac{\mu\pi l^3}{2R^2(\kappa + 1)} - \frac{P_0 l}{4R} - 2w = 0. \tag{6.272}$$

The number of real roots of this equation and the number of stationary points of the function (6.271) depend on the sign of the discriminant

$$\Delta = -(P_0/12Rk)^3 + (w/k)^2, \quad k \equiv \pi\mu/(2R^2(\kappa + 1)). \tag{6.273}$$

If $\Delta > 0$ then we have only one real root

$$l = l_1 = (w/k + \sqrt{\Delta})^{1/3} + (w/k - \sqrt{\Delta})^{1/3},$$

which is positive.

The transition to the limit with $w \to 0$ leads to the Hertz solution:

$$l = \sqrt{P_0 R(\kappa + 1)/(2\pi\mu)}.$$

The transition to the limit with $P_0 \to 0$ defines the residual (after preliminary loading) cohesion zone:

$$l = \sqrt[3]{2w/k}.$$

If $\Delta \leq 0$, then the equation (6.272) has three real roots, two of those correspond to the local minima of the function (6.271). To obtain the unique solution, we use the fact that if $w > 0$ then the contact zone must be not less than the contact zone with $w = 0$. This reasoning (which can be formalized as an additional constraint for the problem) gives

$$\Sigma_c^* \subseteq \Sigma_c^{**}, \tag{6.274}$$

where Σ_c^{**} is the cohesion zone for $w > 0$ and Σ_c^* is the cohesion zone calculated as the limit of Σ_c^{**} for $w \to 0$. This constraint allows us to select a unique root of the equation (6.209), which is positive. Note that the limit solution is the same as for $\Delta \geq 0$.

Some numerical results are shown in Figure 6.6. There is one curve for $w = 0$ and two curves corresponding to the value $w > 0$. If $P_0 \to +\infty$ then two curves $l(P_0)$ for $w > 0$ tend to the curve $l(P_0)$ for $w = 0$.

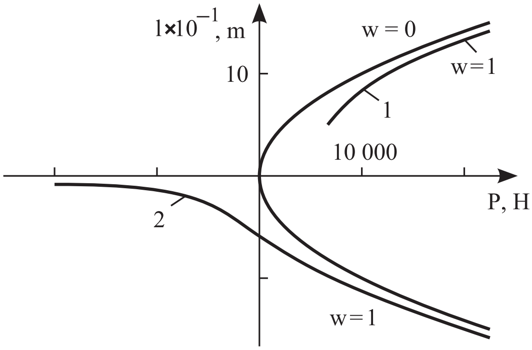

Fig. 6.6. Dependence of the contact force on the contact domain length

The condition (6.274) leads to the statement that the curve 2 is the solution. Note that the nonzero values of l for $P_0 \to -\infty$ are due to the fact that any infinitesimal load on the boundary of the half plane produces an infinite displacements of the boundary points. The numerical results were obtained for the following input data: $E = 10^6$, $\nu = 0.33$, $R = 100$, $w = 1$.

The solution of the contact of two deformed balls was made by K. Johnson [Joh85].

6.4.6 Models of accumulation of damages on the surface

In this section we give examples of the application of theory developed in Sections 6.4.1 and 6.4.2 to the damage accumulation problem. Such models are useful for modeling the crack beginning process at a welded or glued surface. We find also a model which permits us to describe the viscosity of the crack motion (creep). This model will be used later for the contact problem with adhesion, which is not modeled by the JKR theory.

We use the parameter β as a characteristic of the surface damage: the reason for this is that we can consider β as the ratio of the quantity of remaining (undestroyed) bonds per unit area of the surface to the maximum bond quantity in the same surface. We have, for such a choice, $0 \le \beta \le 1$. The parameter β is a function of the coordinates of the surface point and, if we take into account the viscosity destruction, β depends also on the time. The destruction, considered as the beginning of a new free surface, can be of the "threshold" type, i.e., it can start at the moment when the condition $\beta = \beta_* > 0$ is satisfied.

Modeling of damages accumulation without viscosity

We construct a model in which the parameter β changes only if the external loads change, without any retard in time. Consider the problem on the contact of a fixed foundation with the deformed body in the domain Ω (see Section 6.3) and formulate the main hypotheses:

1. The free energy has the form (6.245)
2. The virtual power corresponding to the parameter β is

$$-\int_{\Sigma_c} \varphi(\sigma, \beta)\delta\beta \, \mathrm{d}\Sigma, \tag{6.275}$$

   where the function $\varphi(\sigma, \beta)$ is defined by the contact surface properties and depends on the contact force σ and current value of the parameter β

Repeating the calculation used to obtain the equation (6.231), we find that

$$F_\beta + \varphi(\sigma, \beta) = 0. \tag{6.276}$$

Repeating the reasoning used to obtain the inequality (6.248), we prove that the solution $\{u(x), \beta(x)\}$ of a problem on separation of a deformed (in fact, linear elastic) body from the fixed rigid foundation satisfies the variational inequality

$$\int_\Omega \hat{\sigma}(u) \cdot \cdot \hat{\varepsilon}(\delta u) \, \mathrm{d}\Omega - \int_{\Sigma_c} w\delta\beta \, \mathrm{d}\Sigma \geq -\int_{\Sigma_c} \varphi(\sigma, \beta)\delta\beta \, \mathrm{d}\Sigma + L(\delta u) \quad \forall \delta u, \delta\beta \tag{6.277}$$

with the previous constraints for δu, $\delta\beta$.

Note that at points $x \in \Sigma_c$, for which $0 < \beta < 1$, it follows from the inequalities (6.277) that

$$\varphi(\sigma, \beta) = w, \tag{6.278}$$

and this equation allows us to find the parameter β.

Example 6.13. Consider the problem on the separation of a film from an elastic foundation by a force P, directed along the film. For this problem we obtain the following inequality:

$$\int_{-\infty}^{0} Eu'\delta u' dx - \int_{-\infty}^{x_0} w\delta\beta dx$$
$$\geq -\int_{-\infty}^{x_0} R\delta u dx - \int_{-\infty}^{x_0} \varphi(\sigma, \beta)\delta\beta dx + P\delta u|_{x=0} \quad \forall \delta u, \delta\beta. \tag{6.279}$$

Let the Winkler hypothesis hold, i.e., $\sigma = R = ku$. The simplest approximation of the function φ, which satisfies the requirement of decrease of the parameter β with the increase of $|x|$, can be chosen as

$$\varphi = \varphi(ku, \beta) = \alpha_1 ku + \alpha_2 \beta, \quad \alpha_i = \text{const}, \; i = 1, 2. \tag{6.280}$$

Using (6.280) in the equation (6.278) and taking into account the solution (6.266), we obtain an equation for β, and the solution of this equation as

$$\beta = \begin{cases} [w - \alpha_1 k P e^{\lambda(x-x_0)}/E\lambda]\alpha_2^{-1} \equiv \xi(x), & 0 < \xi < 1, \\ 0, & \xi \le 0, \\ 1, & \xi \ge 1. \end{cases} \tag{6.281}$$

Suppose that

$$\lim_{x \to -\infty} \beta(x) = 1. \tag{6.282}$$

This hypothesis allows us to find the constant $\alpha_2 = w$ of the model. The constants α_1 and w are defined by the properties of the contact surface. Finally, we obtain the solution

$$\beta(x) = \begin{cases} 1 - \dfrac{\alpha_1 k P}{w \tilde{E} \lambda} e^{\lambda(x-x_0)}, & 0 < \xi < 1, \\ 0, & \xi \le 0. \end{cases} \tag{6.283}$$

6.4.7 Model of the viscous crack motion

Governing equations

We analyze first the simplest model of the linear viscosity (the law of “liquid friction”):

$$F_\beta - G_2 = C\dot{\beta}. \tag{6.284}$$

Suppose that a crack grows. We show that the law of liquid friction (6.284) cannot be applied. For this we consider a particular problem on the separation of a thin film from a fixed rigid foundation. Repeating the above reasoning and calculations for (6.284), we find, using the inequality (6.237), that

$$-\int_0^{x_0} w\delta\beta dx \ge \int_0^{x_0} G_2 \delta\beta dx - P\delta x_0. \tag{6.285}$$

By supposition, the foundation is fixed and rigid. Then at the point x_0 of the beginning of the motion we have

$$\beta(x) = 1(x_0 - x), \tag{6.286}$$

where $1(x)$ is the Heavyside function. Then

$$\delta\beta = \delta_0(x_0 - x)\delta x_0, \tag{6.287}$$

where $\delta_0(x)$ is the Dirac delta function. (To distinguish a variation δx_0 from the delta function, we denote the latter by $\delta_0(\cdot)$.)

Using the law (6.231) and equality $F_\beta = G_2$ (there is no dissipation), we obtain that

$$G_2 = -C\dot{\beta}. \tag{6.288}$$

Since

$$\dot{\beta} = \delta_0(x_0 - x)\dot{x}_0, \tag{6.289}$$

then the integral in the right-hand part of the inequality (6.285) diverges and, therefore, the law (6.284) can not be used in a general case.

One model which generalizes the JKR theory and satisfies the physical and mathematical constraints uses the following hypothesis: the irreversible part $F_\beta - G_2$ of the force F_β is a convolution with respect to the spatial variables on Σ_c:

$$F_\beta - G_2 = g * \dot{\beta}, \tag{6.290}$$

which, in a case of a crack of a plane, is written as follows:

$$(g * \dot{\beta})(x) = \iint_{-\infty}^{+\infty} g(x - \xi)\dot{\beta}(\xi)\, d\Sigma_\xi, \tag{6.291}$$

where $g(x) = g(|x|)$ is a fast-decreasing function of the variable $|x|$, e.g.,

$$g(x) = \exp(-c_1|x|^2), \quad g(x) = \frac{c_2}{1 + c_3|x|^{c_4}}. \tag{6.292}$$

This function is determined by an experiment.

Using the potentiality conditions (see Section 3.4), we prove that the force (6.290) and (6.291) corresponds to the potential:

$$U = \frac{1}{2}\langle g * \dot{\beta}, \dot{\beta}\rangle, \tag{6.293}$$

where the angle brackets means the linear functional, e.g., for a plane crack we have

$$\langle g * \dot{\beta}, \dot{\beta}\rangle = \iint_{-\infty}^{+\infty} (g * \dot{\beta})(x)\dot{\beta}(x)\, dx\, dy. \tag{6.294}$$

The generalization of the functional (6.293) for the nonlinear problems, e.g., for a plasticity problem (see, e.g., [Tem86]), introduced by Norton and Hoff, has the form

$$U = \frac{1}{p}\{\langle g * \dot{\beta}, \dot{\beta}\rangle\}^{p/2}. \tag{6.295}$$

Then the irreversible force $F_\beta - G_2$, corresponding to the functional (6.295), is calculated by the formula

$$F_\beta - G_2 = \{\langle g * \dot{\beta}, \dot{\beta}\rangle\}^{p/2-1}(g * \dot{\beta})(x), \tag{6.296}$$

which follows from the definition

$$F - G_2 \in \partial_{\dot{\beta}} U. \tag{6.297}$$

Consider now examples of use of the law (6.296).

Example 6.14. Consider first the problem of separation of a thin film from a fixed rigid foundation. We suppose that just before the separation $\beta = 1$. Then

$$\dot{\beta}(x) = \delta_0(x_0 - x)\dot{x}_0, \quad \delta\beta(x) = \delta_0(x_0 - x)\delta x_0, \tag{6.298}$$

where $\dot{x}_0$ is the speed of the crack tip.

The convolution (6.290) becomes:

$$(g * \dot{\beta})(x) = \int_{-\infty}^{+\infty} g(x-\xi)\dot{\beta}(\xi)\, d\xi. \tag{6.299}$$

Substitution of (6.298) in (6.299) gives the equation

$$(g * \dot{\beta})(x) = g(x - x_0)\dot{x}_0. \tag{6.300}$$

It also follows from (6.298), (6.300) that

$$\langle \dot{\beta}, g * \dot{\beta} \rangle = \dot{x}_0^2 g(0). \tag{6.301}$$

These results, together with (6.296), give the formula for the irreversible force

$$F_\beta - G_2 = \dot{x}_0^{p-1}[g(0)]^{p/2-1} g(x - x_0). \tag{6.302}$$

We suppose that $A = 0$. From the inequality (6.237) we obtain the appropriate variational inequality

$$\int_{-\infty}^{0} \tilde{E}u'\delta u' dx - \int_{-\infty}^{x_0} w\delta\beta dx \geq - \int_{-\infty}^{x_0} ku\delta u dx + \int_{-\infty}^{x_0} G_2\delta\beta dx + P\delta u|_{x=0}. \tag{6.303}$$

We substitute now (6.302) and (6.266) and the variations (6.267) into the inequality (6.303), and suppose that $\dot{x}_0 > 0$. The last supposition means that the inequality (6.303) becomes an equation. Dividing by δx_0, we obtain the following equation for the rate of the motion of the crack tip:

$$\dot{x}_0^{p-1}[g(0)]^{p/2} = w - P^2/2\tilde{E}. \tag{6.304}$$

The solution of this equation for a prescribed external load $P = P(t)$ is the law governing the crack growth $x_0 = x_0(t)$.

Example 6.15. Consider the contact of a round rigid disc with an elastic half-space. We use the results obtained earlier in Section 6.4.5, the variational equation

$$\int_\Omega \hat{\sigma}(u) \cdot \cdot \hat{\varepsilon}(\delta u)\, d\Omega - \int_{\Sigma_c} w\delta\beta\, d\Sigma = L(\delta u) + \int_{\Sigma_c} G_2\delta\beta\, d\Sigma \tag{6.305}$$

following from the definition (6.237), and the same hypothesis as in the JKR theory, excluding only the hypothesis $F = G_2$ instead of which we use the equation

$$F_\beta - G_2 = (\langle g * \dot{\beta}, \dot{\beta} \rangle)^{p/2-1}(g * \beta). \tag{6.306}$$

Repeating the reasoning and calculations which gave the equation (6.302), we obtain the following equation:

$$\int_{-\infty}^{\infty} G_2\delta\beta\, dx = \dot{l}^{p-1}\{2[g(0) + g(2l)]\}^{p/2}\delta l, \tag{6.307}$$

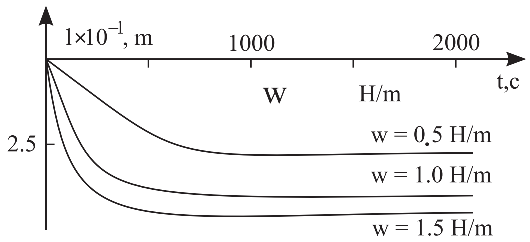

Fig. 6.7. Evolution of the adhesion domain for zero external load

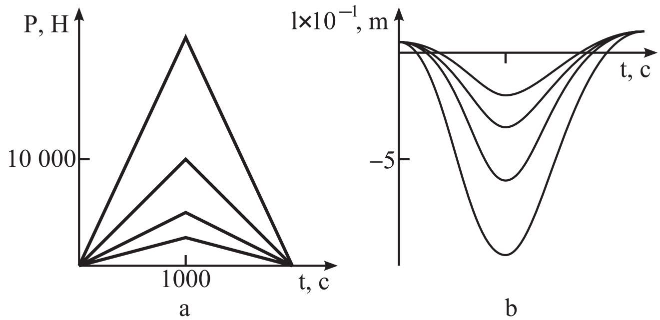

Fig. 6.8. Evolution of the adhesion domain for the different load history

which, after substitution in (6.305), with the formula for the variation δl following from (6.271) and with the formula

$$\delta\beta = [\delta_0(l - x) + \delta_0(l + x)]\delta l, \tag{6.308}$$

gives the equation for the rate of motion of the adhesion zone ends:

$$\dot{l}^{p-1}\{2[g(0) + g(2l)]\}^{p/2} = -\frac{\mu\pi l^3}{2\mathrm{R}^2(\kappa + 1)} + \frac{\mathrm{R}_0 l}{4\mathrm{R}} + 2w \tag{6.309}$$

analogous to the equation (6.304).

Some numerical solutions of the equation (6.309) are shown in Figures 6.7 and 6.8. It follows from an analysis of the curves shown in Figure 6.7 and corresponding to the case $P_0 = 0$ that the adhesion zone monotonically increases, tending to a limit with increase of the growth of the constant w.

Figure 6.8 shows the curves $l(t)$ corresponding to the different processes $P_0(t)$. It can be seen that the system reaction lags the external load $P_0(t)$. This lag increases as P_0 decreases. The input data for computations are the same as in Section 6.4.5: $g(x) = \alpha \exp(-\alpha x^2)$, $\alpha = 1,000$.

Note that, repeating all the previous calculation for the axially symmetric problem on the compression of two balls, first considered by K. L. Johnson [Joh85], we obtain the following equation for the radius a of the adhesion domain:

$$\dot{a}^{p-1}\left[\int_0^{2\pi} d\varphi_\xi \int_0^{2\pi} \tilde{g}(a,a,\varphi_\xi-\varphi_x)\,d\varphi_x\right]^{p/2}$$
$$= E^*\left(-\frac{a^4}{R^2}+a\frac{\delta_1}{R}a^2-\delta_1^2-2\pi a w\right),$$

where $\tilde{g}(\mathrm{r}_x,\mathrm{r}_\xi,\varphi_\xi-\varphi_x)$ is the function in the definition (6.295). For example, if

$$g(|x-\xi|)=\frac{1}{1+|x-\xi|^2}$$

then

$$\tilde{g}(a,a,\varphi_\xi-\varphi_x)=\frac{1}{1+4a^2\sin^2\frac{\varphi_\xi-\varphi_x}{2}},$$

where δ_1 is the relative approach of the balls given as a function of the time. Other notations are the same as in Section 6.4.6.

To conclude this paragraph, we note that there are many studies of crack initiation based on the application of the equation [Che79]

$$\dot{A}+\dot{Q}=\dot{K}+\dot{U}+2\int_l \gamma_0 j\,ds, \tag{6.310}$$

where $\dot{A}$ is the rate of the mechanical work in the system, $\dot{Q}$ is the rate of the heat inflow, $\dot{K}$ is the kinetic energy, $\dot{U}$ is the internal power, γ_0 is a constant of the material referring to the density of the superficial energy, and j is the rate of motion of the crack edge being a spatial curve l in the direction of the outward drawn normal to the contor and in the tangent plane.

The results obtained with the equation (6.310) are considered in more details in the books [Che79, Mor84, Sle02, Hel84] and others.

7

Solution Methods and Numerical Implementation

7.1 Frictionless contact problems: finite element method

7.1.1 Generalities: continuous problem

Recall that in Chapter 4 it was demonstrated that contact problems without friction are equivalent to the minimization of a functional $J(v)$ on the convex subset K of some functional space V:

$$J(v) \longrightarrow \min_{v \in K}, \tag{7.1}$$

where the functional $J(v)$ is given by (4.81) for a contact of a deformed body with a rigid stamp, by (4.107) for the contact of a system of a deformed solids (deformed stamp), and by

$$\begin{aligned} J(v) = \int_0^1 \langle A(tv), v\rangle \, dt =& \frac{1}{2} \int_\Omega [\lambda \theta^2(v) + \mu \varepsilon_{ij}(v) \varepsilon_{ij}(v)] \, d\Omega \\ & - 3\mu \int_\Omega \int_0^{e_u(v)} \omega(s) s \, ds \, d\Omega \equiv J_0(v) - j(v), \end{aligned} \tag{7.2}$$

where

$$j(v) = 3\mu \int_\Omega \int_0^{e_u(v)} \omega(s) s \, ds \, d\Omega, \tag{7.3}$$

for the nonlinear governing equation (3.125).

If $K = V$ (i.e., there are no unilateral constraints) we are dealing with the so-called unconstrained optimization. Then application of the standard gradient method for minimization of the quadratic functionals, (4.81) and (4.107) gives a linear equation (a linear boundary value problem). If in this case the convexity condition holds (the functional J is strictly convex), the solution is unique and is the solution of this linear equation.

If $K \subset V$ then we have the so-called classic "constrained nonlinear programming problem." Methods for its solution are given in [KT51, AHU58, Ban83b, NRKT89] and elsewhere. The bestknown of these methods are:

1. Penalty function methods
2. The gradient projection method
3. The method of centers
4. The cutting plane method (the Kelly method)
5. Barrier-functional methods
6. Methods of feasible directions
7. Duality methods, including the augmented Lagrangian methods (see Chapter 5)

An exhaustive exposition of these (and some other) methods is given in [Céa71, NST06, Ban83a, Fed78, Fle81, HHNL88, KNGK04]. In our study we use two methods: the gradient projection method and the saddle-point search method.

In the gradient projection method the transition from the current iteration number "r" to the iteration "$(r+1)$" is performed in three steps ($u^{(0)}$ is prescribed):

Step 1. Calculate the contact stress $\sigma_N^{(r)}$ corresponding to the approximate solution $u^{(r)}$ and the linear form $L^{(r)}(v) = \langle f^{(r)}, v\rangle$.

Step 2. Solve the linear (for the linear elastic materials) boundary value problem:

$$a(u^{(r+1)}, v) = L^{(r)}(v) \quad \forall v \in V. \tag{7.4}$$

Step 3. If $u^{(r+1)}$ does not belong to K then

$$u^{(r+1)} \longleftarrow P_K(u^{(r+1)}), \tag{7.5}$$

where P_K is the orthogonal projection on the set K.

The saddle-point search method is based on the following variational principle (see Section 5.3):

$$J(v) + \int_{\Sigma_c} \sigma_N(\delta_N - v_N)\, d\Sigma \longrightarrow \sup_{\sigma_N \leq 0} \inf_{v \in V} \tag{7.6}$$

related to the contact of a single solid and rigid stamp.

For the contact problem involving several deformed solids the Young–Fenchel–Moreau (FYM) transformation gives the following saddle-point search problem:

$$J(v) + \sum_I \int_{\Sigma_c^I} \sigma_{NI}(\delta_N - v_N^I + v_N^J)\, d\Sigma \longrightarrow \sup_{\sigma_{NI} \leq 0} \inf_{v \in V}. \tag{7.7}$$

To solve the problem (7.6) or (7.7), a special algorithm is proposed by Uzawa [AHU58]. In this method the nonpositivity restriction on the contact effort σ_N is taken into account by an orthogonal projection onto the set of nonpositive functions. Application of the Uzawa algorithm to the problem (7.6) gives the following iteration process:

Step 1. Choose the zero-approximation for the contact forces $\sigma_N = \sigma_N^{(0)}$.
Step 2. Solve the minimization problem

$$J(v) + \int_{\Sigma_c} \sigma_N^{(0)} (\delta_N - v_N)\, d\Sigma \longrightarrow \inf_{v \in V}, \tag{7.8}$$

which is equivalent to the usual elasticity problem with the following boundary condition on Σ_c:

$$\sigma_{ij} n_j |_{\Sigma_c} = \sigma_N^{(0)} n_i. \tag{7.9}$$

(n_i now denote the components of the outward unit vector orthogonal to Σ.) The result is the displacement field $u^{(1)}$.

Step 3. Calculate the new value of the contact tractions $\tilde{\sigma}_N$ using approximation $u^{(1)}$ for the displacement field.
Step 4. Calculate the orthogonal projection of these tractions onto the set of the nonpositive functions:

$$\sigma_N^{(1)} = \begin{cases} 0, & \text{if } \tilde{\sigma}_N > 0, \\ \tilde{\sigma}_N, & \text{if } \tilde{\sigma}_N \le 0. \end{cases}$$

Step 5. Using some criterion for stopping the iteration process, we finish this process or replace $\sigma_N^{(0)}$ by $\sigma_N^{(1)}$ and go to Step 2.

It is known that this process is convergent for the strictly convex functional $J(v)$ and convex set K. An analogous algorithm can be formulated for several deformed solids in contact. Notice that the direct minimization of the functional (4.81), (4.107), and (3.132) can be performed by the local variation method developed in [Ban83b].

7.1.2 Finite element method: examples

In this and the following sections we give a brief description of the most widespread discretization methods, e.g., the FEM, the BEM, and some sufficient convergence conditions as well. Notice that, in general, the convergence conditions (foundation of the corresponding approximations) are the same as in existence and uniqueness theorems with an additional theorem on the estimates of the approximation error. Note also that there now exist many BEM and FEM methods which can be applied to unilateral constraint problems, see these examples in [CL91, CL96, BS01, ZT00, Bre04, BA93, DL90, Ali02] and many others. Recent results on the error-controlled adaptive finite elements are given in [Ste02, NR04].

Let consider at first the classical internal FEM approximations. The term "internal" means that the approximate solution is in the same space as the exact solution [GLT81]. An example of external approximation is the finite difference methods (FDM) or the "mortar elements" approximation widely investigated recently [FGVH02]. For internal approximation the variational inequality and minimization problem are equivalent, and we will use both formulations. We consider now the variational inequality

$$a(u, v-u) \geq L(v-u) \equiv \langle f, v-u \rangle \quad \forall v \in K,\ u \in K \tag{7.10}$$

where, as usual, $a(u, v)$ denotes the symmetric positive definite bilinear form, $L(v-u) \equiv \langle f, v-u \rangle$ is the continuous linear form on V, $f \in V^*$, and V^* is the conjugate to V.

Let $\{V_h\}$ be a family of finite-dimensional subspaces of V, and $K_h \subset V_h$ be a closed convex set. By definition, the approximate solution of the variational inequality (7.10) is the solution of the following variational inequality:

$$a(u_h, v_h - u_h) \geq L(v_h - u_h) \equiv \langle f, v_h - u_h \rangle \quad \forall v_h \in K_h,\ u_h \in K_h. \tag{7.11}$$

The previous hypotheses on $a(u, v)$ and $L(v)$ in the inequality (7.11) permits us to demonstrate that the inequality (7.11) is equivalent to the minimization problem

$$J(v_h) = \frac{1}{2} a(v_h, v_h) - \langle f, v_h \rangle \longrightarrow \min_{v_h \in K_h} \tag{7.12}$$

Theorem 7.1 (Convergence theorem). *Suppose that*

(i) For all $v \in K$ there exists $\{v_h\} \in K_h$ such that $\lim\limits_{h \to 0} v_h = v$
(ii) For all $\{v_h\}$, $v_h \in K_h$, from $\lim\limits_{h \to 0} v_h = v$ it follows that $v \in K$.

Then

$$\lim_{h \to 0} \|u - u_h\|_V = 0. \tag{7.13}$$

To formulate the convergence conditions for the saddle-point search method, we introduce the notation

$$\mathcal{L}(v, \sigma_N) = \frac{1}{2} a(v, v) - L(v) + \int_{\Sigma_c} \sigma_N (\delta_N - v_N)\, d\Sigma. \tag{7.14}$$

Introduce, in addition to V, a Hilbert space Y and its dual Y^*. Denote an arbitrary element Y by μ, and the solution σ_N of the saddle-point search problem by λ. The functional (7.14) can be written as

$$\mathcal{L}(v, \mu) = \frac{1}{2} a(v, v) - \langle f, v \rangle + b(v, \mu) - [g, \mu], \tag{7.15}$$

where $b(v, \mu)$ is a bilinear form on the direct product $V \bigotimes Y$, and $[g, \mu]$ is a linear form on Y (equals zero in (7.15), but in a more general problem we can have an additional linear form).

Let $K \subseteq V$, $\Lambda \subseteq Y$ be closed nonempty sets, and let $(u, \lambda) \in K \bigotimes \Lambda$ be a saddle point of the functional $\mathcal{L}$, i.e.,

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \mu) \quad \forall (v, \mu) \in K \bigotimes \Lambda. \tag{7.16}$$

If K and Λ are convex sets then the problem (7.16) is equivalent to the system of variational inequalities

$$a(u, v-u) + b(v-u, \lambda) \geq \langle f, v-u \rangle \quad \forall v \in K,\ u \in K, \tag{7.17}$$

$$b(u, \mu - \lambda) \leq [g, \mu - \lambda] \quad \forall \mu \in \Lambda,\ \lambda \in \Lambda. \tag{7.18}$$

Introduce a sequence $\{K_h\}$ of finite-dimensional sets in V (as earlier) and a sequence $\{\Lambda_H\}$ of finite-dimensional sets in Y, $H > 0$.

The approximate solution of the problem (7.16) is defined as an element (u_h, λ_H) satisfying the inequalities

$$\mathcal{L}(u_h, \mu_H) \leq \mathcal{L}(u_h, \lambda_H) \leq \mathcal{L}(v_h, \mu_H) \quad \forall (v_h, \mu_H) \in K_h \bigotimes \Lambda_H. \tag{7.19}$$

The convergence theorem is formulated as follows:

Theorem 7.2. *The limit equalities*

$$\lim_{h \to 0} \|u_h - u\|_V = 0, \tag{7.20}$$

$$\lim_{H \to 0} \|\lambda_H - \lambda\|_Y = 0 \tag{7.21}$$

hold if the following hypotheses (sufficient conditions) are satisfied:

(i) Λ is a convex cone containing a zero element of Y or there exists a number $c_1 > 0$ such that

$$\|\mu_H\|_Y \leq c_1 \quad \forall \mu_H \in \Lambda_H. \tag{7.22}$$

(ii) There exists a constant $c_2 > 0$ independent on h and H such that

$$\sup_{V_h} \frac{b(v_h, \mu_H)}{\|v_h\|_V} \geq c_2 \|\mu_H\|_Y \quad \forall \mu_H \in \Lambda_H. \tag{7.23}$$

(iii) For all $v \in K$ there exists $v_h \in K_h$ such that $v_h \to v$, $h \to 0$ in V.
(iv) For all $\mu \in \Lambda$ there exists $\mu_H \in \Lambda_H$ such that $\mu_H \to \mu$, $H \to 0$ in Y.
(v) From $v_h \in K_h$, $v_h \to v$, $h \to 0$ it follows that $v \in K$.
(vi) From $\mu_H \in \Lambda_H$, $\mu_H \to \mu$, $H \to 0$ it follows that $\mu \in \Lambda$.

To solve the problem (7.19), the finite-dimensional version of the Uzawa algorithm given in Section 7.1.1 can be applied. The FEM is a special procedure for constructing the subspaces V_h, Y_H, and sets $K_h \subseteq V$, $\Lambda_H \subseteq Y$ that satisfy all the hypotheses mentioned above. We give a brief description of this procedure and formulate hypotheses ensuring the convergence of the approximate solution sequence to the exact solution, following Ciarlet [Cia78].

Consider a partition $\Im_h$ of the domain $\bar{\Omega}$ ($\bar{\Omega}$ is the closure of the domain Ω):

$$\bar{\Omega} = \cup T_i, \quad T_i \in \Im_h \tag{7.24}$$

with the following restriction on $\Im_h$: each element $T_i \in \Im_h$ is a polyhedron or nondegenerate polynomial transformation of a polyhedron. For all two elements $T_i \in \Im_h$, $T_j \in \Im_h$, $i \neq j$, there are only three possible cases:

1. $T_i \cap T_j = \emptyset$
2. T_i and T_j have only one common vertex
3. T_i and T_j have only one common edge

Note that such a partition is called a "triangulation" [Cia78].

Let $P = P_k$ be a polynomial space of degree less or equal to k, defined on the subdomain T. This space generates a set $\{V_h\}$ of finite-dimensional subspaces V_h and a family $\{K_h\}$ of admissible elements $v_h \in V_h$. Every subspace V_h is the union of polynomials defined on different T_i. Unification is performed by means of the equations ensuring the continuity requirements for the functions from V_h and (if possible) its derivatives on $\bar{\Omega}$.

The general definition of the finite element is given as follows: A finite element in $\mathbb{R}^n$ is a triple (T, P, S) where

1. T is a closed subset of $\mathbb{R}^n$ with a nonempty interior and Lipschitz-continuous boundary
2. P is a finite-dimensional space of the functions defined over the set T, $N = \dim P$
3. S is a set of N linear forms ϕ_i, $i = 1, \ldots, N$, defined over the space P and satisfying the following requirement: given any scalars α_i, $i = 1, \ldots, N$, there exists a unique function $p \in P$ that satisfies

$$\phi_i(p) = \alpha_i, \quad i = 1, \ldots, N \tag{7.25}$$

This hypothesis permits us to conclude that there exists a base $\{p_i\}_{i=1}^N$ such that

$$\phi_j(p_i) = \delta_{ij}, \quad j = 1, \ldots, N, \tag{7.26}$$

and

$$p = \sum_{i=1}^{N} \phi_i(p) p_i \quad \forall p \in P. \tag{7.27}$$

The linear forms ϕ_i are called the degrees of freedom of the finite element.

Example 7.3. As an example of a set of the degrees of freedom we give

$$\begin{aligned} p &\longrightarrow p(a_i^0), \\ p &\longrightarrow \nabla p(a_i^1)\xi_{ik}^1, \\ p &\longrightarrow \nabla^2 p(a_i^2)(\xi_{ik}^2, \xi_{il}^2). \end{aligned} \tag{7.28}$$

The first case in (7.28) corresponds to an unknown value of the solution at points a_i^0, the second case corresponds to an unknown first derivative in the direction ξ_{ik}^1, and the third case corresponds to a combination of the derivative of the second order at the point a_i^2 which corresponds to the vectors ξ_{ik}^2, ξ_{il}^2.

Note that this definition of a finite element permits us to consider nonpolyhedral partitions of $\bar{\Omega}$, nonpolynomial approximations of the solutions and a sufficiently arbitrary degree of freedom. But the most important features of this approach are as follows:

1. Unification of finite element construction procedures
2. Possibilities for theoretical foundations

The first problem is solved by the choice of a so-called basic element $(\hat{T}, \hat{P}, \hat{S})$ and by use of the mapping F_T of the basic element onto an arbitrary element (T, P, S). In practice, we use, as a rule, the affine mapping

$$F_T : \hat{x} \in \mathbb{R}^n \longrightarrow F_T(\hat{x}) = B_T\hat{x} + b_T, \tag{7.29}$$

where B_T is a nondegenerate matrix and b_T is a vector. Both depend on T.

For the theoretical foundation, introduce the so-called P-interpolate of the function $v \in V$:

$$\Pi v = \sum_{i=1}^{N} \phi_i(v)p_i. \tag{7.30}$$

The advantage of the affine mapping consists in the conservation of the polynomial type for the P-interpolate on T.

It can be demonstrated that the error of such an interpolation involves the norms of the matrices B_T and B_T^{-1}, the last depending on the parameter h being equal to the diameter of the domain T [Cia78]. The convergence theorem is formulated as follows:

Theorem 7.4. *The limit equality*

$$\lim_{h \longrightarrow 0} \|u - u_h\| = 0 \tag{7.31}$$

holds with an additional hypothesis: the angles at the vortex must be bounded from below when $h \to 0$.

We now give some results regarding the solution to contact problems. All these solutions were found with a polynomial of degree "1" for the displacement fields and piece-wise constant approximations of the contact forces σ_N on the contact surface Σ_c. In the next example, V is a subspace of $H^1(\Omega)$, Y is a subspace of $L_2(\Sigma)$, K is defined (as usually) with the impenetrability condition, and Λ is the cone of nonpositive functions. All the convergence conditions hold.

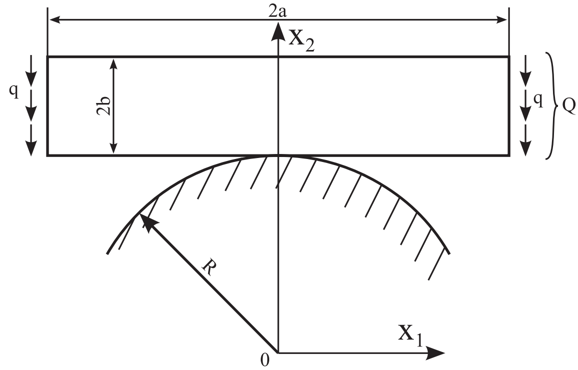

Fig. 7.1. Contact of the deformed rectangle with the rigid stamp

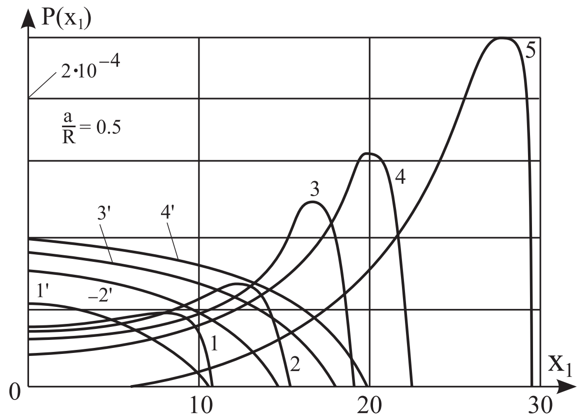

Fig. 7.2. Distributions of the contact pressure for the different external forces

Example 7.5. Consider a 2D contact problem for the rectangle and rigid fixed stamp (see Figure 7.1). External actions are the shear forces uniformly distributed on the lateral sides of the rectangle.

The distributions of the contact stress $\sigma_N \equiv P$ are given in Figure 7.2 (distances are measured in cm, pressures in kg/cm^2). The different curves correspond to different values of external load – an increase in q corresponds to an increase in the number of curves. Broken curves correspond to the Hertz contact theory. The Hertz solution is acceptable for small values of external load only.

It is well known that if we use the Kirchhoff beam theory to model the deformation of a quadrilateral domain, then the contact efforts are reduced to two concentrated forces [Gal48] (continuous distribution occurs only in an exceptional case). It can be seen that the obtained solutions tend to two concentrated forces when the external forces increase.

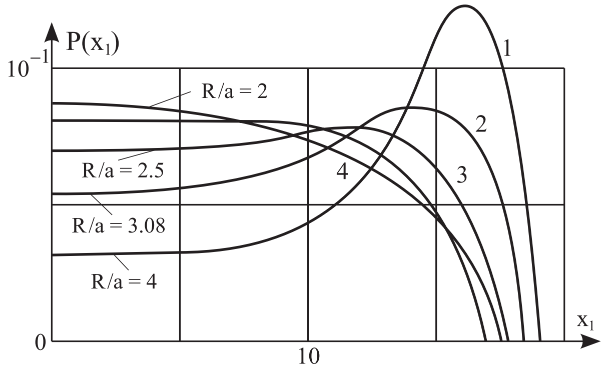

Fig. 7.3. Distributions of the contact pressure for different relations R/a

The dependence of the distributions of the contact pressure on the relation R/a is given in Figure 7.3, the external load being fixed. We can see that the solution tends to the Hertz solution when R/a increases.

As expected, the discrepancy between the FEM and Hertz solutions increases when the external load increases.

Note also that the majority of these results were obtained with the Uzawa method. To estimate the effectiveness of this method with respect to other methods, we also used the projection gradient method. The correspondence between two approximate solutions was found good, but more CPU time was required for the projection gradient method than for the Uzawa method. This is why we use the Uzawa method.

Example 7.6. Consider an axially symmetric contact problem for a hollow cylinder or cone and a rigid cylindrical shell (see Figure 7.4). The upper end wall is stress-free and the lower end wall slides without friction on the plane foundation. External loads are the internal pressure and volume forces parallel to the Oz-axis with the density ρF. In the calculations $\rho F = 0.1$. Other input data (in relative units) is: the Young modulus $E = 300$, the Poisson ratio $\nu = 0.47$, $p = 0$. Relative sizes are given in Figure 7.4.

The contact pressure distributions are given in Figure 7.5. The curve I corresponds to the hollow cylinder with $\delta = 0.05$, the curve II corresponds to the conic shell with the angle of taper 0.23 rad and $\delta = 0.05$ at the upper point, and the curve III corresponds to the conic shell with the angle of taper 0.23 rad and $\delta = 0.05$ at the lower point. One can see that the control exerted by the form of the shell permits decrease in the maximum of the contact pressure.

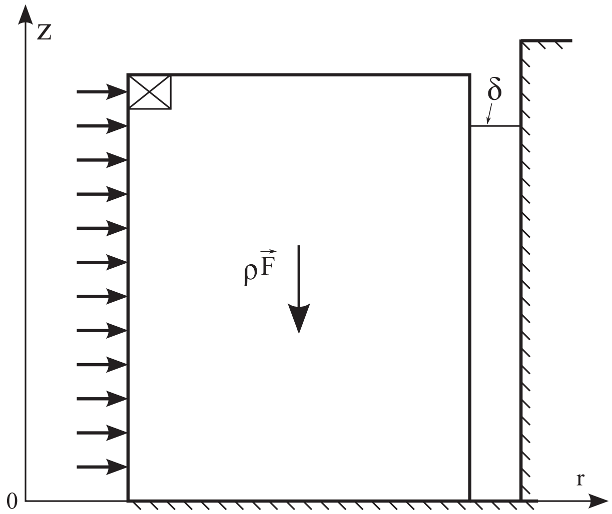

Fig. 7.4. Axially symmetric contact problem for a hollow cylinder or cone in a shell

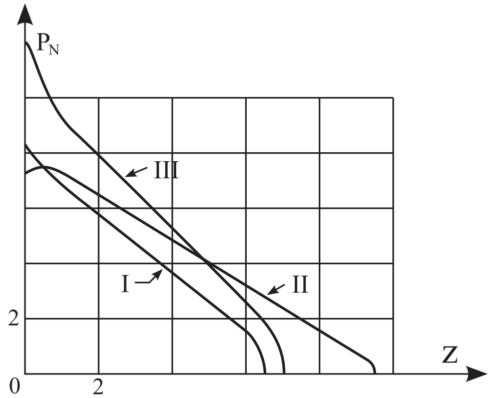

Fig. 7.5. Distributions of the contact force in the axially symmetric problem

7.2 Friction contact problems: boundary element method

7.2.1 Boundary element method

We give a brief description of BEM for the linear theory of elasticity. In the calculations two versions of BEM were used. The first one corresponds to the direct formulation using the fundamental solution of the Lamé equations, i.e., to the Green function for an infinite space. In the second variant the Boussinesq and Cerruti solutions for a half-space are used.

BEM with the fundamental solution of the Lamé equations

Consider the following integral equation corresponding to the BVPs of the linear theory of elasticity [BA93]:

$$c_{ij}(\xi)u_j(\xi) + \int_\Gamma p^*_{ij}(\xi,x)u_j(x)\,d\Gamma_x \\ - \int_\Gamma u^*_{ij}(\xi,x)p_j(x)\,d\Gamma_x + \int_\Omega u^*_{ij}(\xi,x)b_j(x)\,d\Omega_x = 0, \quad (7.32)$$

where Γ is the boundary of the domain Ω, coefficients c_{ij} are defined by the properties of the curve Γ, b_i are the components of the given volume forces density, and for 3D problems $i,j = 1,2,3$. In this subsection we construct numerical solutions for 2D problems. For these problems the following changes in the indices were made: $i \leftarrow \alpha$, $j \leftarrow \beta$, $\alpha,\beta = 1,2$. The index "j" will be used to number the boundary elements and their ends.

The kernels in the integral equation (7.32) for the isotropic body are equal to

$$p^*_{\alpha\beta}(\xi,x) = -\frac{1}{4\pi(1-\nu)r}\left\{\left[(1-2\nu)\delta_{\alpha\beta} + 2\frac{\partial r}{\partial x_\alpha}\frac{\partial r}{\partial x_\beta}\right]\frac{\partial r}{\partial n} \right. \\ \left. -(1-2\nu)\left(\frac{\partial r}{\partial x_\alpha}n_\beta - \frac{\partial r}{\partial x_\beta}n_\alpha\right)\right\}, \quad (7.33)$$

$$u^*_{\alpha\beta}(\xi,x) = -\frac{1}{8\pi G(1-\nu)}\left[(3-4\nu)\ln r\delta_{\alpha\beta} - \frac{\partial r}{\partial x_\alpha}\frac{\partial r}{\partial x_\beta}\right], \quad (7.34)$$

where n_α are the components of the outward unit vector orthogonal to Γ, $\delta_{\alpha\beta}$ is the Kronecker symbol, and

$$r = r(\xi,x) = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}, \quad x \in \Gamma,\ \xi \in \Gamma.$$

The constants c_{ij} in (7.32) depend on the smoothness of the boundary at the point ξ and are calculated simultaneously with the calculation of the matrix of the BEM system equations.

Introduce a partition of Γ onto N_E boundary elements $\Gamma_j, j = 1,2, \ldots, N_E$, using a linear approximation of the displacements and pressures on Γ_j. To take into account possible discontinuities of the pressure on Γ, we use the double-knot conception which consists of the different approximations in the neighboring boundary elements. Suppose that the boundary element Γ_j is a rectilinear segment and introduce a local coordinate η, $-1 \le \eta \le 1$, and basic functions on $[-1,+1]$:

$$\varphi_1 = (1-\eta)/2, \quad \varphi_2 = (1+\eta)/2. \quad (7.35)$$

Enumerate the ends of the element Γ_j (called "knots" in BEM) by the integers j – the left end, $(j+1)$ – the right end. We are proceeding along the curve

Γ such that the domain Ω is on the left. Denote the segment $[x^{(j)}, x^{(j+1)}]$ by Γ_j. With the basic functions (7.35) can be written as

$$r|_{\Gamma_j} = (\varphi_1 x_1^j + \varphi_2 x_1^{j+1}, \varphi_1 x_2^j + \varphi_2 x_2^{j+1}). \tag{7.36}$$

The set $\{\Gamma_j\}_{j=1}^{NE}$ describes the curve Γ exactly when Γ is a polygon. Otherwise an additional error appears which must be taken into account in the convergence investigation [DL90].

The same approximation is used for the boundary displacements and efforts:

$$u|_{\Gamma_j} = (\varphi_1 u_1^j + \varphi_2 u_1^{j+1}, \varphi_1 u_2^j + \varphi_2 u_2^{j+1}), \tag{7.37}$$

$$p|_{\Gamma_j} = (\varphi_1 p_1^j + \varphi_2 p_1^{j+1}, \varphi_1 p_2^j + \varphi_2 p_2^{j+1}). \tag{7.38}$$

After partition of the curve Γ into the elements Γ_j, we obtain (we assume that $b_\alpha = 0$)

$$c_{\alpha\beta}(\xi)u_\beta(\xi) + \sum_{j=1}^{N_E} \left[\int_{\Gamma_j} p^*_{\alpha\beta}(\xi, x)u_\beta(x)\, d\Gamma_x - \int_{\Gamma_j} u^*_{\alpha\beta}(\xi, x)p_\beta(x)\, d\Gamma_x \right] = 0. \tag{7.39}$$

Using the approximation of the boundary Γ by the rectilinear segments Γ_j and the approximations (7.35)–(7.37), after transformation of all the integrals in the integral over segment $[-1, +1]$ with the independent variable η, we obtain the following system of linear equations:

$$c^{ij}_{\alpha\beta} u_\beta(\xi^{(i)}) + \sum_{j=1}^{N_E} \left\{ \frac{l_j}{2} \int_{-1}^{1} [G]_{ij}(\eta)\, d\eta \right\} u^{(j)} - \sum_{j=1}^{NE} \left\{ \frac{l_j}{2} \int_{-1}^{1} [H]_{ij}(\eta)\, d\eta \right\} p^{(j)} = 0, \tag{7.40}$$

where $i = 1, 2, \ldots, 2 \cdot N_E$ is the number of equations (for each knot $\xi^{(i)}$ on Γ there are two equations), $[G]_{ij}$, $[H]_{ij}$ are $2 \bigotimes 4$-matrices, and the quantities

$$u^{(j)T} = (u_1^j, u_2^j, u_1^{j+1}, u_2^{j+1}), \quad p^{(j)T} = (p_1^j, p_2^j, p_1^{j+1}, p_2^{j+1})$$

represent the set of unknowns for the boundary element number "j." Here "T" denotes the transposition operator.

Note that if a classical boundary condition holds at each knot $\xi^{(j)}$ on Γ then at each knot we have two unknown values u_1^j, u_2^j or p_1^j, p_2^j or their combination and two equations. Then the corresponding systems of the linear equations are closed.

We shall also consider piece-wise homogeneous structures. In this case at the knots on the discontinuity surface we have four unknown values u_1^j, u_2^j, p_1^j, p_2^j for each homogeneous piece. At a common knot there are two equations related to each body in contact. To close the system of linear equations, we add the continuity conditions

$$p^+_\alpha(\xi^{(j)}) = p^-_\alpha(\xi^{(j)}), \quad u^+_\alpha(\xi^{(j)}) = u^-_\alpha(\xi^{(j)}) \tag{7.41}$$

or identify the unknowns for the two sides of the bodies in contact.

7.2.2 Numerical examples to 2D contact problems

We suppose that the stamp (indenter) is moving, and rewrite all the equations, statements, and formulae of Section 4.3.6 for the 2D quasistatic contact problem for a deformed body and a rigid moving rough stamp, introducing the notations $\mathbf{x}, \mathbf{u}, \ldots$ for vector fields.

After deformation the position of arbitrary points $\mathbf{x} \in \Sigma_C$ is $\mathbf{x} + \mathbf{u}(\mathbf{x}, t)$. Then, taking into account the formula (4.193) and using the above hypotheses on the function Ψ, we have the first constraint at the surface Σ_C:

$$\Psi\{[A]^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p)\} \geq 0 \quad \forall \mathbf{x} \in \Sigma_C, \tag{7.42}$$

which is the impenetrability requirement. Recall that

$$\sigma_N(\mathbf{x}, t) \leq 0 \quad \forall \mathbf{x} \in \Sigma_C \tag{7.43}$$

and that

$$\Psi\{[A]^{-1} \cdot (\mathbf{x} + \mathbf{u}(\mathbf{x}, t) - \mathbf{U}_p)\}\sigma_N(\mathbf{x}, t) = 0 \quad \forall \mathbf{x} \in \Sigma_C. \tag{7.44}$$

To close the set of relations on the part Σ_C of the boundary, we must describe the model for the tangential contact pressure. If there is no friction then

$$\boldsymbol{\sigma}_T(\mathbf{x}, t) = 0 \quad \forall \mathbf{x} \in \Sigma_C. \tag{7.45}$$

We will use the Coulomb friction law:

$$|\boldsymbol{\sigma}_T| \leq f|\sigma_N| \implies \dot{\mathbf{u}}_T = 0, \tag{7.46}$$
$$|\boldsymbol{\sigma}_T| = f|\sigma_N| \implies \exists \kappa \geq 0 : \dot{\mathbf{u}}_T = -\kappa \boldsymbol{\sigma}_T \tag{7.47}$$

with the quantity $\dot{\mathbf{u}}_T$ being the relative slip velocity of the boundary points of a deformed body, i.e., is the derivative of the relative displacements with respect to the parameter t.

Variational setting and the solution method

First, we transform the local setting to the variational one for a moving stamp using the results of Section 4.3.6. A difference with the inequality (4.170) consists of the new definition of the kinematically admissible velocities. The main result for the quasistatic problem can be formulated as follows:

Theorem 7.7. *The quasistatic friction contact problem is equivalent to the variational inequality:*

$$a(\mathbf{u}, \delta\dot{\mathbf{u}}) + \int_{\Sigma_c} f|\sigma_N(u)|(|\dot{\mathbf{v}}_T| - |\dot{\mathbf{u}}_T|)\, d\Sigma \geq L(\delta\dot{\mathbf{u}})$$
$$\forall \delta\dot{\mathbf{u}} = \dot{\mathbf{v}} - \dot{\mathbf{u}},\ \ \dot{\mathbf{v}} \in \dot{K}_u,\ \ \dot{\mathbf{u}} \in \dot{K}_u,\ \ \mathbf{u} \in K, \tag{7.48}$$

where

$$a(\mathbf{u},\delta\mathbf{u}) = \int_{\Omega} \hat{\sigma}(\mathbf{u}) \cdot\cdot \hat{\varepsilon}(\delta\mathbf{u})\, d\Omega, \tag{7.49}$$

$$L(\delta\mathbf{u}) = \int_{\Omega} \rho\mathbf{F}\cdot\delta\mathbf{u}\, d\Omega + \int_{\Sigma_\sigma} \mathbf{P}\cdot\delta\mathbf{u}\, d\Sigma, \tag{7.50}$$

$$\dot{K}_u = \{\dot{\mathbf{v}} \mid \dot{\mathbf{v}} = \dot{\mathbf{u}} + \delta\dot{\mathbf{u}};\ \Psi(\boldsymbol{\alpha})'_{\alpha}\cdot(A^{-1}\cdot\delta\dot{\mathbf{u}}) \geq 0,\ \forall \mathbf{x}\in\Sigma_C^t\}, \tag{7.51}$$

$$\Sigma_C^t = \{\mathbf{x} \mid \mathbf{x}\in\Sigma_C;\ \Psi(\boldsymbol{\alpha}(\mathbf{x})) = 0;$$

$$[A]'\cdot([\dot{A}]^{-1}\cdot(\mathbf{x}-\mathbf{U}_p+\mathbf{u}) + [A]\cdot(-\dot{\mathbf{U}}_p+\dot{\mathbf{u}})) = 0\}, \tag{7.52}$$

$$[A] = [A](\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} = [A]^{-1}\cdot(\mathbf{x}-\mathbf{U}_p+\mathbf{u}). \tag{7.53}$$

The index "′" denotes the derivative with respect to the variable $\boldsymbol{\alpha}$, the set K is defined as earlier. The inequality (7.48) formally is the same as the analogous inequality in Section 4.3. The newness is in the definition of admissible set $\dot{K}_u$.

The proof of this theorem is based on the proofs of the analogous theorems in [Kra80, Kra97, KNG04] reproduced in Section 4.3. Note that the obtained inequality belongs to the set of the quasi-variational inequalities. It is appropriate for theoretical analysis.

The solution method is based on the idea of the Uzawa algorithm (as in Section 4.3.4). First, such an algorithm was proposed in [DL72] to a problem of minimization of the nondifferentiable functional

$$J(v) = \frac{1}{2}\int_{\Omega} \varepsilon(\mathbf{v})\cdot\cdot{}^4\hat{a}\cdot\cdot\varepsilon(\mathbf{v})\, d\Omega - \int_{\Sigma_\sigma} \mathbf{P}\cdot\mathbf{v}\, d\Sigma + \int_{\Sigma_C} f|\mathcal{F}||\mathbf{v}_T|\, d\Sigma$$
$$- \int_{\Sigma_C} \mathcal{F} v_N\, d\Sigma \equiv \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + j_T(\mathbf{v}) - j_N(\mathbf{v}), \tag{7.54}$$

where the forms $a(\mathbf{v},\mathbf{v})$, $L(\mathbf{v})$ are defined by (4.63) and (4.64), and

$$j_T(\mathbf{v}) = \int_{\Sigma_C} f|\mathcal{F}||\mathbf{v}_T|\, d\Sigma, \quad j_N(\mathbf{v}) = \int_{\Sigma_C} \mathcal{F} v_N\, d\Sigma, \tag{7.55}$$

$\mathcal{F}$ being a known function which can be the contact pressure.

We now formulate a theorem which allow us to construct an algorithm for the friction problem solution.

Theorem 7.8. *The following limit equality holds:*

$$f|\mathcal{F}||\mathbf{v}_T| = \max_{\boldsymbol{\mu}_T,\, \boldsymbol{\mu}_T\leq f|\mathcal{F}|} [\boldsymbol{\mu}_T\cdot\mathbf{v}_T]. \tag{7.56}$$

In principle, the statement (7.56) is trivial. The scalar product of two vectors, one of which is a constant and the other is changing inside a fixed circle, attains the maximum where the variable vector is on the circumference and is parallel to a constant vector. But when we deal with generalized functions, the demonstration can be difficult, see, e.g., the monograph [Cia88].

Using Theorems 7.7 and 7.8, we prove that the minimization problem for the functional (7.54) is equivalent to the following saddle-point problems:

$$\frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + \int_{\Sigma_c} (\mu_T \cdot \mathbf{v}_T)\, d\Sigma - j_N(\mathbf{v}) \longrightarrow \min_{\mathbf{v}\in V} \max_{|\mu_T|\le f|\mathcal{F}|} . \qquad (7.57)$$

The stationarity conditions for the functional (7.57), i.e., that the functional derivatives of the functional (7.57) are zero, yield $\boldsymbol{\mu}_T = -\boldsymbol{\sigma}_T$.

The problem (7.57) can be solved by any method for the saddle problem search. We will use the Uzawa method (see [GLT81] and Section 7.1.1). This method is the alternating movement to the saddle point in the direction of the fastest decrease with respect to the variable $\mathbf{v}$ and the fastest increase with respect to the variable $\boldsymbol{\mu}_T$ of the functional (7.57). If some step violates the imposed restrictions, we return to the admissible set by the shortest way.

We modify the problem (7.57) by the change of the function $\mathcal{F}$ to an unknown contact pressure σ_N. We suppose also that there exists a gap δ_N between the contacting body which depends on the applied load. If, as in (7.54), the function σ_N is known then we obtain the following saddle-point problem:

$$\frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + \int_{\Sigma_c} [\mu_T \cdot \mathbf{v}_T + \sigma_N(\delta_N - v_N)]\, d\Sigma \longrightarrow \min_{\mathbf{v}\in V} \max_{|\mu_T|\le f|\mathcal{F}|} . \qquad (7.58)$$

We now suppose that the contact pressure is an unknown function and that the problem (7.58) for such a case is modified as follows:

$$\frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + \int_{\Sigma_c} [\mu_T \cdot \mathbf{v}_T + \sigma_N(\delta_N - v_N)]\, d\Sigma \longrightarrow \min_{\mathbf{v}\in V} \max_{\sigma_N\le 0} \max_{|\mu_T|\le f|\sigma_N|} . \qquad (7.59)$$

An argument for such hypotheses is that for one step in a loading process the velocities in the Coulomb friction law can be changed to displacements. Additionally, the results of Chapter 5 allow us to take into account a gap as in (7.59).

The last modification consists of the hypothesis that a displacement increment $d\mathbf{u}^t = \mathbf{u}^{t+1} - \mathbf{u}^t$ corresponding to the transition from t to $t + dt$ satisfies

$$\frac{1}{2}a(\mathbf{v},\mathbf{v}) - L^{(t+dt)}(\mathbf{v}) + \int_{\Sigma_C} [\mu_T \cdot (\mathbf{v}_T - \mathbf{u}_T^t) + \sigma_N(\delta_N - v_N)]\, d\Sigma \longrightarrow \min_{\mathbf{v}\in V} \max_{\sigma_N\le 0} \max_{|\mu_T|\le f|\sigma_N|} . \qquad (7.60)$$

The Uzawa method for the problem (7.60) contains the following steps (comp. with the Uzawa method for the contact problem without friction, see Section 7.1.1):

Step 1. Choose a zero-approximation $\sigma_N^{(0)}$, $\boldsymbol{\sigma}_T^{(0)}$ for the contact stresses.

Step 2. Find the minimum of the functional (7.60) with respect to the displacements $\mathbf{v}$. This step is equivalent to the solution of the usual (without constraints in the form of inequalities) BVPs of linear elasticity with the boundary conditions on the surface Σ_c:

$$\sigma_{ij}\nu_j|_{\Sigma_c} = \sigma_N^{(0)}\nu_i + (\boldsymbol{\sigma}_T^{(0)})_i. \tag{7.61}$$

As a result we have the displacement field $\mathbf{u}^{(0)}$.

Step 3. Using this displacement field, we calculate the corresponding contact stress distributions

$$\sigma_N^{(1)} = P_N(\sigma_N^{(0)} + \rho_{0N}(\delta_N - u^{(0)})), \tag{7.62}$$

$$\boldsymbol{\sigma}_T^{(1)} = P_T(\boldsymbol{\sigma}_T^{(0)} + \rho_{0T}(\mathbf{u}_T^{(0)} - \mathbf{u}_T^t)), \tag{7.63}$$

where

$$P_N(\sigma_N) = \begin{cases} \sigma_N, & \sigma_N \le 0 \\ 0, & \sigma_N > 0 \end{cases} \tag{7.64}$$

$$P_T(\boldsymbol{\sigma}_T) = \begin{cases} \boldsymbol{\sigma}_T, & |\boldsymbol{\sigma}_T| \le f|\sigma_N^{(0)}| \\ \dfrac{\boldsymbol{\sigma}_T}{|\boldsymbol{\sigma}_T|} f|\sigma_N^{(0)}|, & |\boldsymbol{\sigma}_T| > f|\sigma_N^{(0)}| \end{cases} \tag{7.65}$$

are the orthogonal projection operators of the corrected contact stresses on the admissible set of contact stresses defined – as was pointed out above – by the zero adhesion stresses and the Coulomb friction law. ρ_{0N}, ρ_{0T} are the numerical parameters controlling the convergence rate.

Step 4. Go to Step 2 with the new values of the contact stresses.

Note that we obtain the same algorithm, using the initial Lagrange variational equation (4.180), generalized onto the friction contact problem with a moving stamp, and using a heuristic idea (as in Section 4.3.4) to satisfy all the inequality constraints.

Discretization

We give here the discretization only with respect to the parameter t which defines the history of the loads. We find the solution $\mathbf{u}(\mathbf{x}, t)$ for $t \in [0, T]$. Let a partition of the segment $[0, T]$ be given by the points (knots) t_k, with $t_0 = 0$, $t_N = T$, $\Delta t_k = t_{k+1} - t_k$. Let the solution be $\mathbf{u}^k$ at the point $t = t_k$ and $\mathbf{u}^{k+1}$ at the point $t = t^{k+1}$. We suppose that the initial state $\mathbf{u}^0$ is prescribed.

Usually, $\mathbf{u}^0$ is chosen as being the natural state – the state without strains and stresses. Then, to solve the problem for $t = t_{k+1}$, we obtain the quasi-variational inequality

$$a(\mathbf{u}^{k+1}, \mathbf{v} - \mathbf{u}^{k+1}) - L(\mathbf{v} - \mathbf{u}^{k+1}) + \int_{\Sigma_c} f|\sigma_N(\mathbf{u}^{k+1})|(|\mathbf{v}_T - \mathbf{u}_T^k| - |\mathbf{u}_T^{k+1} - \mathbf{u}_T^k|)\, d\Sigma \geq 0 \quad \forall \mathbf{v} \in K,\ \mathbf{u}^{k+1} \in K, \quad (7.66)$$

which is solved with the iteration method proposed above.

We use the BEM for discretization in space (see Section 7.2.1).

Numerical examples and analysis

First, we test the algorithm and computer code. To do this, we compare an analytical solution and the numerical one. Note that analytical solutions of the model friction contact problems are in [KO88, Gor98]. The friction contact problem with stick and slip domains was considered first in [Spe75].

It was found in the numerical tests that the number of iteration in the proposed method is less than in the FEM, due to the fact that the equilibrium equation and boundary conditions on Σ_σ and on Σ_C are satisfied exactly and errors appear as interpolation errors, integration errors, and errors of an approximation of the boundary.

Comparison of the numerical and analytical solutions

First consider the embedding of a rigid rectangle into the half plane [Mus53, Section 117–117a]. An analytical solution to this problem was obtained for the friction law

$$\sigma_T = f\sigma_N \quad (7.67)$$

valid for the constant sign of σ_T in the contact domain. f is the friction coefficient. Choose the axis Ox of a Cartesian coordinate system along the boundary of the half plane, and direct the axis Oy inside the half plane. The solution for the embedding without rotation and horizontal translation of the stamp has the form

$$\sigma_N = P_0/[\pi\sqrt{(x-a)(x+a)}] \quad (7.68)$$

in the contact domain. a is the half-length of the contact domain divided by the half-length of the rectangle (as the variable x) and P_0 is the vertical loading force. The results presented in Figure 7.6 were obtained for two values of a: two curves at the middle of Figure 7.6 correspond to $a = 0.2$ and to $AB = BC = CD = DA = 2$, and two other curves correspond to $a = 0.8$, $AB = CD = 1$, $BC = DA = 2$ (see Figure 7.8). Other input data is: $f = 0.2$, number of the boundary elements is equal 100 for all the sides of rectangle, the total number is 400.

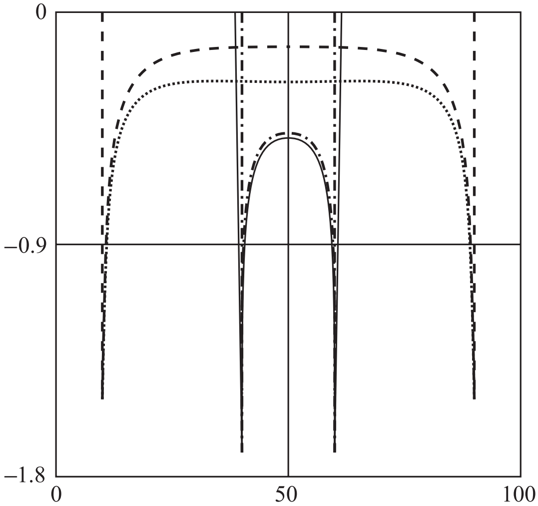

Fig. 7.6. Distributions of the contact pressure

In the first case one curve was obtained by the algorithm as described above and the second curve is the solution (7.68). In the second case the pointwise curve was obtained by the iteration algorithm and the dashed one is the solution (7.68). It can be seen that if the contact domain and the embedding depth are sufficiently small with respect to the rectangle dimensions, then the analytical and numerical solutions are almost the same. In the second case the two solutions differ considerably at the center of the contact domain. Note that the contact pressure for the Coulomb friction law and for the law (7.67) coincide up to about 1%.

An analytical solution for the relative size domain C/L (C is the length of the stick domain, L is the length of contact domain) was obtained in [Gal48, Spe75]. To show the stick and slip domain, we calculate the horizontal displacement u_x on BC for $AB = BC = CD = DA = 2$. The results are shown in Figure 7.7.

The curves "1", "3", "5" correspond to the values $U_y = 0.02, 0.06, 0.1$ of the embedding U_y, $f = 0.1$. It can be seen that the result of the works [Gal48, Spe75], where the relative values of the stick domain does not depend on the vertical pressing force (or embedding depth), is confirmed. The curve "5′" corresponds to the value $f = 0.8$. The numerical value $(C/L)^*$ of the stick domain coordinate for $f = 0.1$ is approximatively 0.56. For $f = 0.8$ we obtain $(C/L)^* \approx 0.92$. These values correspond to those given by the solutions [Gal48, Spe75]. We cannot give the exact value for the numerical value $(C/L)^*$, because it varies smoothly near the ends of the stick domain.

Consider the embedding (without rotation) of a circle into a rectangle (see Figure 7.8) and compare the numerical solution with the analytical one.

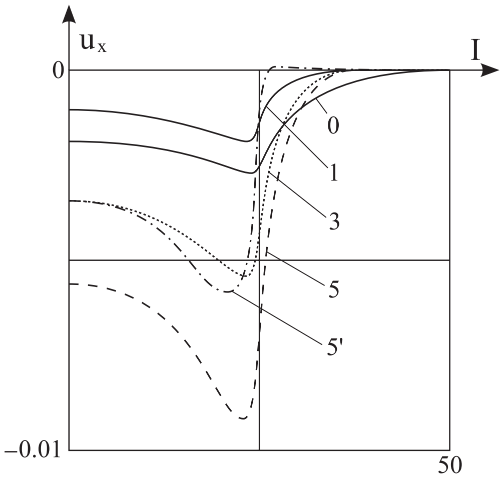

Fig. 7.7. Distributions of horizontal displacement. The curve $5'$ corresponds to the displacement in the Coulomb friction law

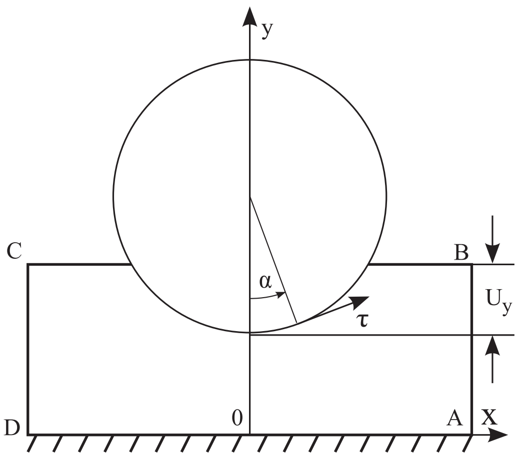

Fig. 7.8. Contact problem scheme

The results shown in Figure 7.9 correspond to the data: $AB = CD = 1$, $BC = DA = 2$, 50 boundary elements at the sides AB and CD, 200 elements at BC and DA, $f = 0.2$, radius of the stamp $R = 4$, embedding $U_y = 0.1AB$. The solid curve corresponds to the analytical solution taken from [Mus53]. The dashed curve corresponds to the numerical solution with five steps in load. It can be seen that two solutions differ from each other by about 2% at the center of the contact domain.

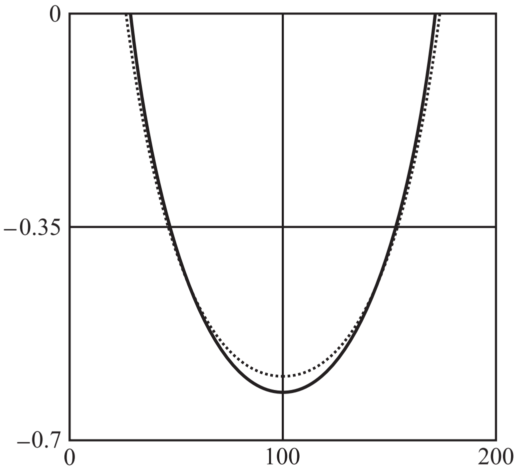

Fig. 7.9. Distributions of the contact pressure for the circular stamp

Numerical convergence

As was mentioned in Chapter 4, there are a number of convergence problems. The first one is the convergence with respect to spacial discretization. The theoretical background is given, e.g., in [GLT81]. The second problem consists of the convergence demonstration with respect to time (loading parameter). Theoretical investigations and an example are given in [RP01]. Note that the numerical tests show that we obtain the stabilization of the numerical results with about 50–100 boundary elements on each side of the designed rectangle and with about 5–10 steps of the loading parameter.

Some numerical results corresponding to the embedding of the rectangle with $a = 0.8$ and with all the previous input data for this value (see above) are presented in Figure 7.10. The difference between the two successive iterations at the end of the calculation is 10^{-7}. The solid curve refers to the value of the normal contact stress in the middle of the contact domain and the dashed curve corresponds to the friction stress multiplied by 5 at the knot in $0.25BC$. The number of iteration is shown on the horizontal axis. It can be seen that the results stabilize after about 50–70 iterations.

New solutions

Let us consider the contact problem for the rigid circular stamp embedded into a deformed rectangle and after embedding rotated with respect to its center (see Figure 7.8). As earlier, we suppose that the side DA of the rectangle $ABCDA$ is fixed, i.e., this segment is the part Σ_u. The part Σ_σ is the union of sides AB and CD. The surface forces on these segments are zero. The part Σ_C is the segment BC.

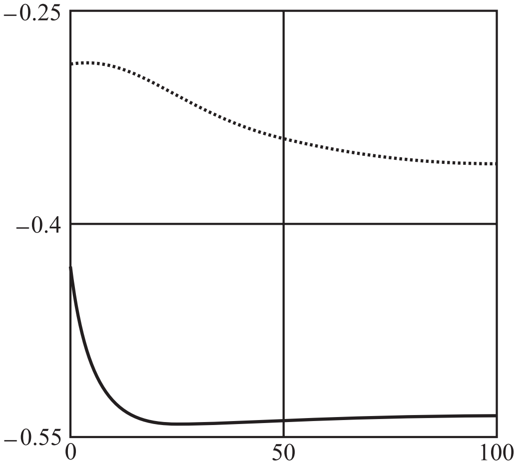

Fig. 7.10. Stabilization of the numerical solution with the increment of iterations

The stamp moving along the axis Oy is embedded into the rectangle. The center of the circle is at the axis Oy, and if there is no rotation, then the problem is symmetric with respect to the axis Oy. Thus, for the embedding step in the condition (7.42) we have

$$\hat{A} = \hat{A}^{-1} = \hat{\delta}, \quad \mathbf{U}_p = (0, U_y), \quad y = AB, \tag{7.69}$$

and the impenetrability condition takes the form

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{u}, \mathbf{U}_p) &= \Psi(((x,y),(u_x,u_y),(U_x,U_y))_{\Sigma_C} \\ &= (x+u_x)^2 + (U_y + u_y - R)^2 - R^2 \geq 0. \end{aligned} \tag{7.70}$$

For the rotation step we have the additional tangent velocity $\Delta\alpha/\Delta s$, which is used for the correction of the tangent contact forces in the Uzawa algorithm, as

$$\boldsymbol{\sigma}_T^{(r+1)(s+\Delta s)} = P_T(\boldsymbol{\sigma}_T^{(r+1)(s)} + \rho_{0T}(\mathbf{u}_T^{(r+1)(s)} - \mathbf{u}_T^t - \boldsymbol{\tau} R \Delta\alpha)), \tag{7.71}$$

where the projection operator is defined by

$$P_T(\boldsymbol{\sigma}_T) = \begin{cases} \boldsymbol{\sigma}_T, & |\boldsymbol{\sigma}_T| \leq f|\sigma_N^{(0)}|, \\ \dfrac{\boldsymbol{\sigma}_T}{|\boldsymbol{\sigma}_T|} f|\sigma_N^{(0)}|, & |\boldsymbol{\sigma}_T| > f|\sigma_N^{(0)}|. \end{cases}$$

R is the radius of the stamp, ρ_{0T} is a numerical parameter for the control of the convergence, and $\boldsymbol{\tau}$ is the vector of the tangent (see Figure 7.8). Note that the increment Δs is in the iteration parameter ρ_{0T}.

We consider the plane stress problem. The calculations were performed with the following input data:

- The number of the boundary elements is 50 on the sides AB and CD and 200 on the sides BC and DA
- Contact stresses in all the figures normalized by dividing by $0.2E$, E is the Young modulus, the Poisson ratio is 0.3
- The maximal value of the stamp displacement U_y along the axis Oy is $0.1AB = 0.05BC$, $AB = 1$, the step on s is 0.02 for embedding (five steps) and 0.001 rad for rotation (five steps, too), the friction coefficient $f = 0.2$.

Note that in the most works on the contact problems the authors use the traditional Hertz formulation for the impenetrability condition. It is supposed, in the such formulation, that the boundary points of the deformed bodies are moved parallel to the unit vector orthogonal to the common tangent plane at the initial contact point. For the problem presented in Figure 7.8 the Hertz impenetrability condition is

$$-R + U_y + u_y + \sqrt{R^2 - x^2} \geq 0. \tag{7.72}$$

The theoretical investigation of the accuracy of such a linearization and for other forms of the impenetrability condition was performed first in [Kra78]. The numerical estimates are given in the examples (see below).

It is important to note that the strong impenetrability condition (7.42) permits the modeling of the impenetrability requirement at the angular points of the boundary. Sometimes, the relative velocities in the Coulomb friction law are replaced by the relative displacements. The friction contact problems with the relative velocities were first solved in [Kra80] for axially symmetric problems, for a ball imbedded in an elastic half-space. This approach permits to investigate a complex (nonproportional) loading of the deformed body. Below we compare the solutions for one and for several steps of the parameter t, which is equal to the embedding depth U_y of the stamp, one step corresponds to the relative displacements in the friction law and several steps correspond to an approximation of the velocities.

The horizontal displacements at the side BC for two formulations of the frictionless contact problem are for

1. The Hertz impenetrability condition (7.72) with the projection of the corrected contact stresses onto the axis Oy
2. The condition (7.42) with the projection of the corrected contact stresses onto the normal to stamp (inside the circle) shown in Figure 7.11

It can be seen that the Hertz formulation of the contact problem gives the understated values for the horizontal displacements. Note that we obtain the same results for the formulation 1 of the contact problem with the condition (7.42), the difference is in the number of iteration for a fixed precision. Note also that the vertical displacements and normal pressure are almost the same for all the considered cases.

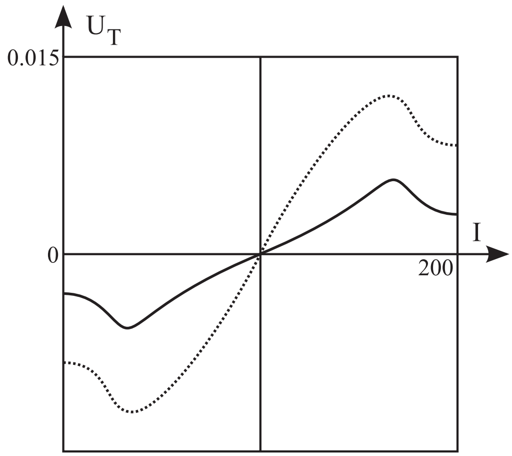

Fig. 7.11. Comparison of the horizonal displacements: solid curve corresponds to the case 1, dotted curve corresponds to the case 2. I is the number of the element from B to C

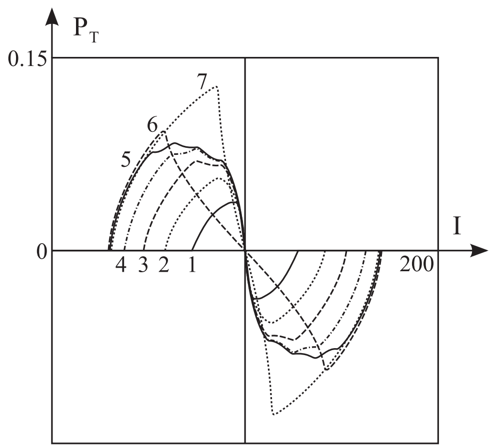

Fig. 7.12. Evolution of the friction contact stresses

The most important results concern the choice of the projection of the corrected contact stresses, i.e., the using the case 1 or the case 2 for the friction contact problem. We investigate also the influence of the loading step and mutual influence of the normal pressure and friction stress on the distribution of the friction stress. Obtained results are collected in Figure 7.12.

The curves 1–5 correspond to five values of the loading parameter $t = U_y = 0.02, 0.04, \ldots, 0.1$. These curves were obtained with the formulation 2 of the friction contact problem. The curves 6 and 7 were obtained for the formulation 1 without taking into account of the mutual influence of the normal pressure and friction stress. The curve 6 corresponds to the completely

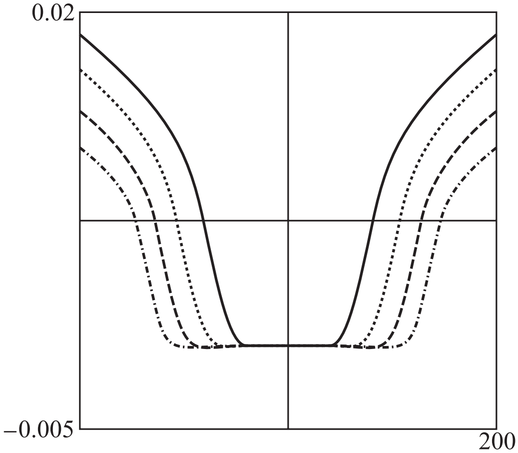

Fig. 7.13. Evolution of the adhesion domain

classical formulation of the friction contact problem, i.e., to one loading step and to the projections onto the axis Oy. The curve 7 is obtained with five loading steps. We see that the difference between the results at the end of loading can be of order 25%.

To investigate the evolution of the stick domain, we calculate the differences of the displacements obtained for the second and first loading steps, for the third and second steps, etc. The obtained curves are shown in Figure 7.13. We see that the horizontal parts of these curves, being the stick domains, increase monotonically with the increase of the parameter t. The most important conclusion is that the maximal values of the friction stresses correspond to the ends of the stick domains.

We now consider the problem of the rotation after embedding with the same input data as in the previous problem, i.e., the number of the boundary elements is 50 at the sides AB and CD and 200 at the sides BC and DA. In all the figures contact stresses are normalized by dividing by $0.2E$, E is the Young modulus, the Poisson ratio is 0.3. The maximal value of the stamp displacement U_y along the axis Oy is $0.1AB = 0.05BC$, $AB = 1$. The steps on s is 0.02 for embedding (five steps) and 0.001 rad for rotation (five steps, too). The friction coefficient $f = 0.2$.

Some numerical results are shown in Figures 7.14 and 7.15. The curve for embedding $U_y = 0.02$ in Figure 7.14 is obtained for the beginning of embedding. All the other correspond to the final value $U_y = 0.1$ of the embedding. The curve “1” is obtained for the fifth embedding step. Note that for 10 embedding steps this curve is almost the same, it becomes more smooth.

The maximal (minimal) values of the friction stresses of the curves in Figure 7.14 correspond to the ends of the stick domains and, respectively, to the initiation of slip. The increase of the number of the steps in time

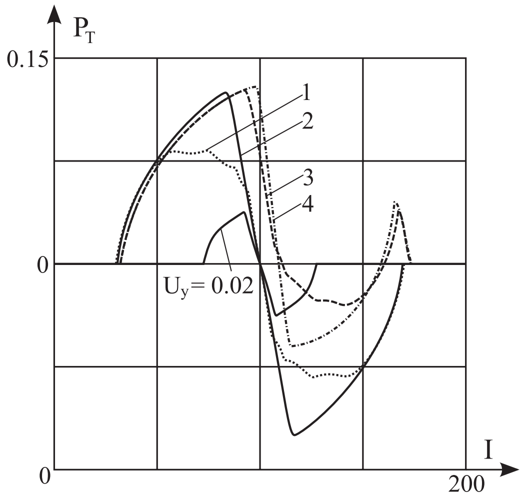

Fig. 7.14. Distributions of the friction stresses for the embedding and rotation: "1" is obtained with five embedding steps equal to 0.02; "2" is obtained with one embedding step equal to 0.1; "3" corresponds to the final value $\alpha = 0.005\,\mathrm{rad}$ of the angle of rotation with five identical rotation steps; "4" is obtained with one step in embedding and consequent one rotation step

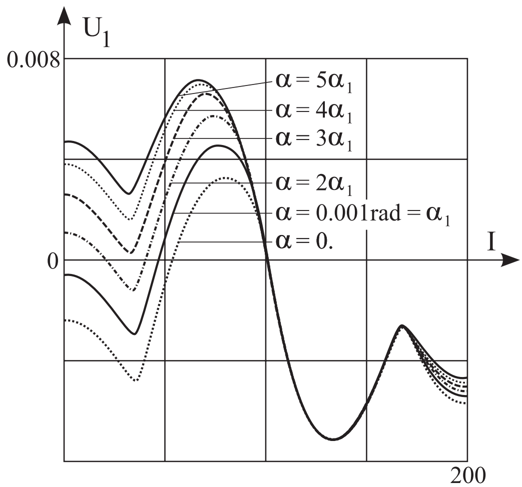

Fig. 7.15. Distributions of the horizontal displacements for six steps of rotation

implies an increase of the curve smoothness at the end of the stick domain. We can see that the rotation implies an asymmetry of the contact stresses and displacements.

The curves presented in Figure 7.15 correspond to six steps of the rotation, value of the rotation angle is pointed in the figure. The confluence of two consecutive curves means that the coincidence domain is the stick domain. These domains move to the right. It can be seen that the increment of the rotation angle implies an increment of the minimal value (negative) of the relative tangential displacement.

7.2.3 Solution to 3D contact problems with the Boussinesq and Cerruti formulae

Choose the origin O of the Cartesian coordinate system $Ox_1x_2x_3 \equiv Oxyz$ at the boundary of half-space and axis Oz being orthogonal to the boundary. Recall that the Boussinesq solution relates the displacements u_3 of the half-space boundary with the concentrated force orthogonal to the plane boundary of the half-space and that the Cerruti solution relates the displacements of the boundary with the concentrated force tangent to the plane boundary of the half-space.

Write these dependences in the form

$$\mathbf{u}(\tilde{\mathbf{x}}) = \mathbf{L}(\tilde{\mathbf{x}}, \mathbf{x})\{\mathbf{p}(\mathbf{x})\}, \tag{7.73}$$

where $L(\tilde{\mathbf{x}}, \mathbf{x})\{\mathbf{p}(\mathbf{x})\}$ is the linear integral operator on the contact stresses $\mathbf{p}$, see, e.g., [Now70] for the formulae.

Note at first that for the contact problem without friction it is sufficient to use the Boussinesq solution only. For generality (and for the contact problem with friction), suppose that the tangential stresses on the boundary of the half-space are not zero.

Let the maximal contact domain be included in a square $Q = \{(x, y) \mid -1 \leq x \leq +1,\ -1 \leq y \leq +1\}$, and let the center of Q be the point O and the side of the square be parallel to the axes Ox_1, Ox_2.

Introduce a partition of Q into a set of subdomains T. Let the boundary element $T = T_{ij}$ be a square with the tops (knots of the BEM):

$$x_1^i = -1 + (i-1)h, \quad x_2^j = -1 + (j-1)h, \quad 1 \leq i, j \leq (M+1), \tag{7.74}$$

where h is the length of the side of a boundary element. To formulate the finite-dimensional approximation of a contact problem, we must calculate the displacements at all the knots. To do this, we suppose that the vector of the boundary tractions $\mathbf{p}$ is constant at an element of the partition of Q and integrate the dependence (7.73) over this element.

Let "ij" be a pair of integers i, j which are the knots number (with the coordinates (7.74)) and let T_{ij} be an element where the knot (x_1^{ij}, x_2^{ij}) is the lower left top. Denote the displacements of the knot "ij" by the vector or column $\{U^{ij}\} = \{u_1^{ij}, u_2^{ij}, u_3^{ij}\}^T$ and the stresses, being, by supposition, constants on the element T_{kl} by $\{P^{kl}\} = \{p_1^{kl}, p_2^{kl}, p_3^{kl}\}^T$. Introduce the set of matrices $[A_{ij}^{kl}]$ relating the displacements of the knot "ij" and stresses on the element "kl":

$$[A_{ij}^{kl}] = \begin{bmatrix} 2(1-\nu)I_1 + 2\nu I_2 & 2\nu I_3 & (-1+2\nu)I_4 \\ 2\nu I_3 & 2(1-\nu)I_1 + 2\nu I_5 & (-1+2\nu)I_6 \\ (-1+2\nu)I_4 & (-1+2\nu)I_6 & 2(1-\nu)I_1 \end{bmatrix},$$

where

$$\begin{aligned} I_1 &= \int_{T_{kl}} R^{-1}\, d\xi_1\, d\xi_2, & I_2 &= \int_{T_{kl}} R^{-3}(x_1-\xi_1)^2\, d\xi_1\, d\xi_2, \\ I_3 &= \int_{T_{kl}} R^{-3}(x_1-\xi_1)(x_2-\xi_2)\, d\xi_1\, d\xi_2, & I_4 &= \int_{T_{kl}} R^{-2}(x_1-\xi_1)\, d\xi_1\, d\xi_2, \\ I_5 &= \int_{T_{kl}} R^{-3}(x_2-\xi_2)^2\, d\xi_1\, d\xi_2, & I_6 &= \int_{T_{kl}} R^{-2}(x_2-\xi_2)\, d\xi_1\, d\xi_2. \end{aligned} \tag{7.75}$$

Note that all these integrals are calculated precisely.

Using the superposition principle, we obtain

$$\{U^{ij}\} = \sum_{k,l} [A_{ij}^{kl}]\{P^{kl}\}. \tag{7.76}$$

This formula gives the solution to the boundary value problem for the half-space with the prescribed surface stress.

Examples

Consider the contact problem for a rough ball, cylinder, and cube and a linear elastic half-space. To solve the boundary value problem of linear elasticity with the boundary condition (7.61) on Σ_c, we use the formula (7.76).

Example 7.9. Consider a rigid rough ball imbedded in an elastic half-space without rotation. First, we investigated the same problems as in Section 7.2.2, i.e., the influence of the impenetrability condition and number of loading steps on the normal and tangential contact stresses and displacements. It was found that quantitatively this influence is similar to that obtained to 2D contact problems.

The most interesting results concern the tangential contact stresses, i.e., friction stresses. The 3D distribution of the function $\boldsymbol{\sigma}_T(x, y, 0)$ is shown in Figure 7.16. The domain for calculation of the matrix $[A_{ij}^{kl}]$ is the rectangle $-1 \le x, y \le +1$ with the mesh of 50×50 elements. The illustrated contact stresses were normalized by dividing by $0.2E$, E is the Young modulus, the Poisson ratio is 0.3, and the ball radius $R = 2$.

Note that the maximal values of the function stress $\boldsymbol{\sigma}_T(x, y, 0)$ correspond to the end of the stick domain and, therefore, to the beginning of the slip domain. This problem was solved as a 3D problem, but, in fact, it is a 2D problem if we use the cylindrical system of coordinates.

The most interesting results for 3D formulation were obtained for the distributions of the relative slip velocities for the rolling of a ball on the plane

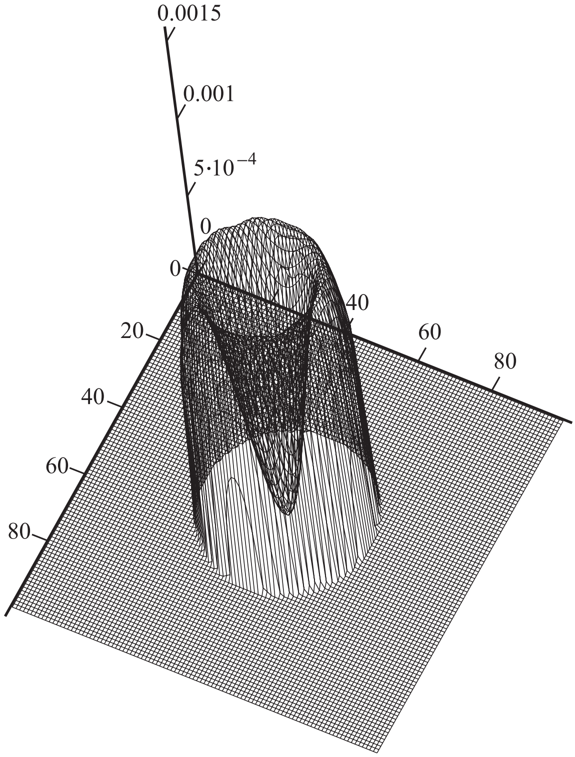

Fig. 7.16. Distribution of the modulus of friction stresses at the boundary of half-space

boundary of an elastic half-space. In particular, it was found that there exists a domain where the particle slides forward (ahead of movement) and a domain where it slides back.

Example 7.10. We now consider the embedding of a rigid cylinder of length L into the elastic half-space. This problem was investigated for the same input data as above. The axis of cylinder is parallel to the axis Ox, its radius is equal 2.0, and the length is 1.6. The normal pressure is shown in Figure 7.17 and the modulus of the friction stress for the friction contact problem in Figure 7.18.

It can be seen that there are a concentration of the contact stress near the ends of the cylinder. There exist two maximums of the friction stresses corresponding to the end of the stick domain.

Example 7.11. Let now a rigid cube of the length L of its edge be embedded into the elastic half-space. Suppose that $L = 1.6$ and all the rest input data is the same as in the previous examples. The normal pressure is shown in Figure 7.19. Note that this quantity is practically the same in the frictionless problem and in the friction contact problem – we cannot differ them in the figure.

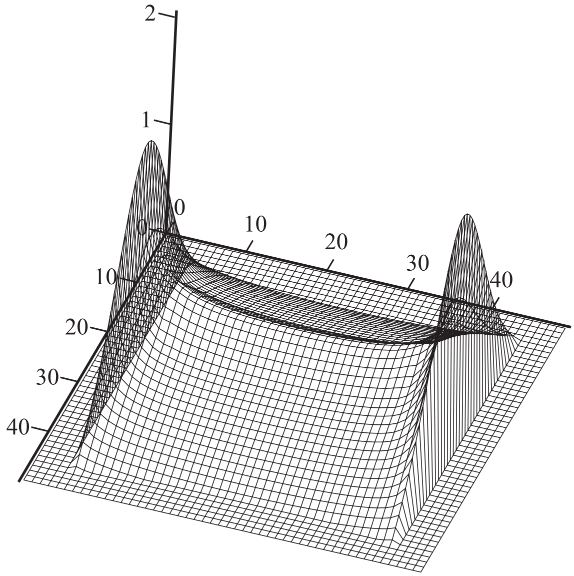

Fig. 7.17. Distribution of the normal pressure at the boundary of half-space

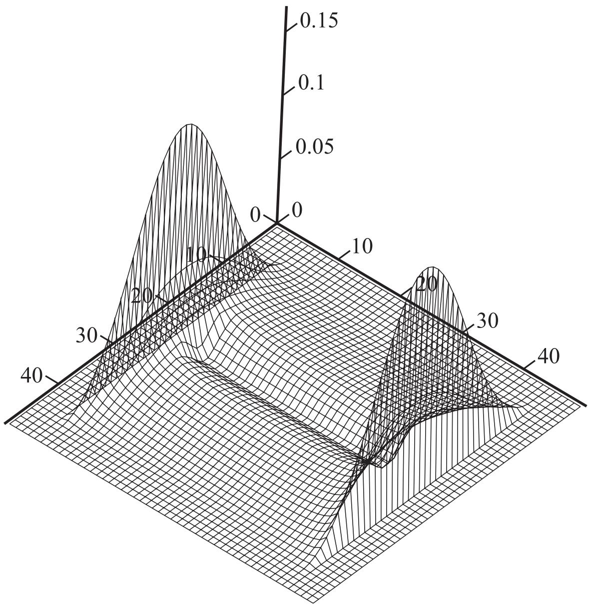

Fig. 7.18. Distribution of the friction stresses at the boundary of half-space

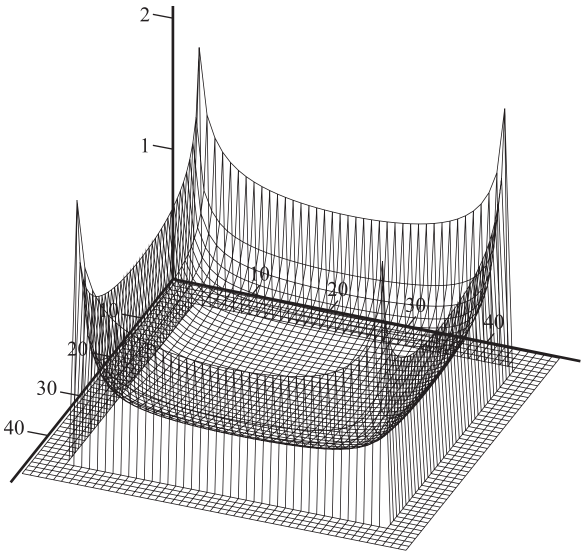

Fig. 7.19. Distribution of the normal pressure for the friction contact problem at the boundary of half-space

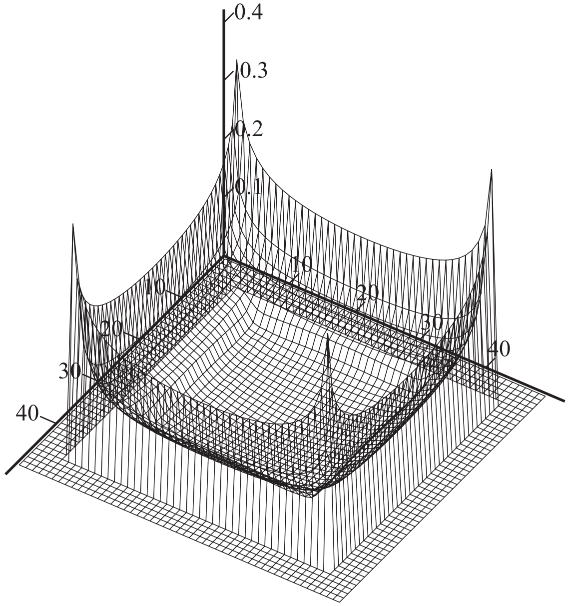

Fig. 7.20. Distribution of the normal pressure for the friction contact problem at the boundary of half-space

We can see that there is a concentration of the contact stresses near the edges and at the tops of the cube. The concentration near the tops is stronger than near the edges. Therefore, it can be reasonable to suppose that in an analytical solution the singularities at the tops will be stronger than at the edges.

The normal displacement is shown in Figure 7.20. Note that there is no gap between the face of cube and boundary of half-space for the considered embedding (= 0.1). All the results were obtained for five loading steps and the result for 10 steps are the same as in the figures.

There are fewer publications on the BEM foundation than for analogous problems in FEM.

The classical theory for singular integral equations was created in the works of S.G. Miklin, see [Mik64a, Mik65]. Applications to some 2D contact problem performed with the complex variable, the Kolosov–Muskhelishvili potentials and the corresponding integral equations are described in [Mus53, Mik64a, Mik65, Gla80]. These works concern analytical solution of the equations derived from the classical mixed boundary problems without any unilateral constraints. An approach to the problems with unilateral constraints based on the variational method is described in [DL90].

8

Concluding Remarks

The unilateral problems and variational methods for solving them, described in this book, have a wide range of practical applications and possibilities for further development, and the scope of this book has allowed us to treat just a few of them. To compensate for this lack, we cite significant current lines of research in this field.

8.1 Modeling, and identification problem, and optimization

The modeling (mathematical modeling) consists of two essential steps. The first deals with the construction of the equations and conditions describing the considered system or process. These equations and conditions include constants and functions which characterize the physical properties and depend, in general, on the relevant spatial and/or temporal variables.

Secondly, we must find all these constants and functions using experimental data. This step, which is the most important, is usually called *identification* and reduces the solution to an *inverse problem.*

We describe the general modeling scheme which is used, entirely or partly, in practice. This scheme is shown in Table 8.1 for an example of ultrasound wave propagation modeling. The number of experiments depends on the number of parameters to be determined and can be, theoretically, infinite.

The term “verification” (see Table 8.1) means that we apply the identified parameters to the solution of problems which differs from those solved in the identification step (Step VI). Solutions found at this step must be validated by complementary experiments. If the theoretical predictions are in good agreement with the experimental data, then the mathematical model is considered to be adequate and it may be used in applications.

If the theoretical and experimental results differ significantly, we must return to Step III or even to Step I to improve the model by replacing the state variable and/or the characteristics of the internal structure.

Table 8.1. General modeling scheme

Step	General scheme	Parameters and equations
I	Choice of variables for characteristic object or process under study	Acoustics: pressure $P = P(x,t)$ material density $\rho = \rho(x,t)$ velocity of particles $v = v(x,t)$
II	Choice of equations reflecting conservation	Motion equations: Euler form $-\nabla P = \rho \dfrac{dv}{dt}$ Mass conservation law $\dfrac{d\rho}{dt} + \rho \nabla \cdot v = 0$
III	Choice of state, internal structure, or process characteristics	Initial distribution of material density: $\rho_0 = \rho_0(x)$ volume modulus of elasticity: $K = K(x)$
IV	Construction of governing equations	Barotropy equation: $P = F(\rho)$
V	Transformation of equations to forms describing the particularities of external actions, e.g., linearization	Linearized barotropy law: $P = K(\rho - \rho_0)$ Linearized motion equation: $-\nabla P = \rho_0 \dfrac{\partial v}{\partial t}$ Linearized mass conservation law: $\dfrac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0$
VI	Identification of internal structure parameters	Determination of parameters by comparison $\rho_0 = \rho_0(x)$, $K = K(x)$ of the experimental data and theoretical solutions
VII	Model verification	Comparison of the theoretical solutions and experimental data for the problems different of those used in the identification step

Consider in detail Step VI, i.e., parameter identification under the hypothesis that the model describing the object or process is known. Consider some of the features in the treatment of experimental data relating to contact problems arising from nondestructive testing of a specimen or structure.

Let a model be represented by the boundary value problem for the differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + cu = f(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{8.1}$$

with boundary conditions

$$l_1(u(x)) = u_{1,\Sigma}(x), \quad x \in \Sigma = \partial\Omega, \tag{8.2}$$

$$l_2(u(x)) = u_{2,\Sigma}(x), \quad x \in \Sigma = \partial\Omega, \tag{8.3}$$

where Ω is an open-bounded domain in the Euclidean space $\mathbb{R}^n$ with the smooth boundary $\Sigma = \partial\Omega$ and operators $l_i(\cdot)$ define the boundary conditions. The coefficients $a_{ij}(x)$ and the operator $l(\cdot)$ satisfy ellipticity conditions. If we use one of the boundary conditions (8.2) which may be, e.g., a Dirichlet or Neumann condition, then this hypothesis ensures the existence of a solution and permits us to describe the steady state of an object or process.

The physical characteristics of the object under investigation define the coefficients $a_{ij} = a_{ij}(x)$, $b_k(x)$, $c(x)$. External actions on the object and the response of the body are defined by the functions $f(x)$, $u_{i,\Sigma}(x)$, $i = 1, 2$. The identification problem, and an essential step in the modeling, is the determination of the functions $a_{ij} = a_{ij}(x)$, $b_k(x)$, $c(x)$ provided that we know the functions $f(x)$, $u_{i,\Sigma}(x)$.

Generally, such a problem has no solution. To explain this statement, let us consider the case $n = 3$. In this case, the boundary data given by the functions $u_{i,\Sigma}(x)$ depends on just two variables governing the boundary variety Σ. At the same time the coefficients of the equation (8.1) depend on three space variables. Suppose that all these functions are expressed by only one function $\sigma(x)$ which describes some physical property of the body under investigation.

We thus have two functions to determine, $u(x)$, $\sigma(x)$, depending on three spatial variables. It is impossible, in general, to find two functions dependent on the three variables from two functions which depend on two variables only.

This problem can be solved by introducing one (or several) additional parameter. Let this parameter (independent variable) be a scalar denoted by s. Then

$$u_{i,\Sigma} = u_{i,\Sigma}(x, s), \quad x \in \Sigma = \partial\Omega. \tag{8.4}$$

Parameter s is defined by the experimental conditions. Introduction of this parameter allows us to identify at least one internal parameter.

A problem of determining a differential equation coefficient from certain given information is known as an *inverse coefficient problem* [TA79].

The problem of determining the right-hand side $f(x)$ in the differential equation (8.1) is called an *inverse source problem.* An example of such a problem is the determination of the coordinates of the electrical potential source on an epicardium surface (heart surface). Similar problems arise in theoretical seismology in which the source of the seismic waves is unknown.

A special feature of both problems is their ill-posedness [TA79]. The state function $u = u(x)$ as well as the coefficient to be defined are unknown. Therefore, the identification problem is nonlinear.

Usually, nonlinear problems are solved by iterative methods. One of them consists of the solution of the boundary value problems (8.1) and (8.2) with the coefficients $a_{ij}, b_k, \ldots$ obtained from the previous iteration (the initial values of coefficients are to be assigned). These kinds of problems are called *direct problems.* Therefore, the identification problem requires the solution of a sequence of direct problems.

In the mechanics of solids, instead of the single equation (8.1), we are dealing with a system of differential equations for a vector-function, integral or integro-differential equations, or with a system of equations. If the experimental data are collected using a test of the contact type, then the boundary condition in the mathematical model contains inequalities describing an impenetrability requirement, zero adhesion, etc. (see Chapter 4).

Corresponding quasi-static problems on identification of the elastic modulus for a heterogeneous body were first formulated and solved in [Con95]. Experimental data in contact type tests are considered in [CT00]. The uniqueness problem is investigated in [MY04, BG04].

It should be emphasized that the variational approach to the solution of identification problems means taking into account new mechanical and geometrical constraints for the identified parameters in the variational inequalities.

Methods for the identification of the mechanical properties for homogeneous fully anisotropic bodies are proposed in [Yak90].

Dynamic identification problems in the linear theory of elasticity were investigated in [KN03a] and methods for solving direct problems arising in the identification procedure are given in [KN03b]. The solution methods for inverse problems in vibration are developed in [Gla04].

Note that methods for solving contact problem are similar to those developed for the solution of optimization problem. Indeed, it is well known that the formulation of an optimization problem includes functional (or functionals in the multicriteria optimization) minimization under unilateral and bilateral constraints. Therefore, the situation arises in contact problems of reducing the problem to one of functional minimizations. Methods developed especially for optimization are described in [NRKT89, Ban83a, HA79, Pra74] and many others. There are also works devoted to the optimization of structures in unilateral contact [Pet95, HN88].

8.2 Development of the contact problems with friction, wear, and adhesion

In the theory and solution of contact problems with friction two essential formulations are used. The first continues the approach proposed by Duvaut and Lions [DL72] for friction laws with displacements. (The foundation for this simplification was given in Section 4.3.) A considerable contribution to the mathematical theory of this friction law is given in [Pan85].

The general mathematical theory of variational (hemivariational) inequalities related to contact problems was developed in [NP95]. The method to quasi-variational inequalities solution which reduce the problem to a set of the Wiener–Hopf equations is proposed in [Din98].

The essential difficulty in variational inequality solutions stems from the nondifferentiability of the terms which describe the friction phenomena. In most studies, the nondifferentiability is overcome with the formula (7.56) (see Chapter 7). An example of the research in this direction can be found in [MM02, Ste02] and many others.

Friction laws resembling those for plastic flow are proposed in [MM78, HC93, HdSM02].

Modern studies have been devoted to more widely applicable friction laws, including the relative sliding velocities of superficial particles. Methods for solving quasistatic and stationary rolling problems are developed in [Kal66]. Examples of time discretization in dynamic contact problems can be found in [PKM$^+$02, CMEAR01, AKS97, RKMO02]. An existence theorem of the weak solution for a dynamic friction contact problem is given in [KS04] using a penalization and regularization technique. Analogous problem is considered in [IN02] for a viscoelastic body, with the hypotheses like to those used in Sections 4.3.4 and 4.3.5.

A survey of the traditional mathematical models in which the wear of the contacting surfaces is taken into account is given in [Gor98]. Note that there is (almost) no theoretical research on the wear phenomenon using the variational approach. Applications to the rail-wheel interaction problem can be found in [Esv01].

Adhesion phenomena plays an important role in micromechanics and nanomechanics. Physically consistent models of adhesion including nonlocal laws of contact interaction were described in Chapter 6. As was pointed out in this chapter, it is possible to use the local adhesion theory if the scale of the model is greater than a few interatomic distances [JKR71]. An example of the investigation and numerical solution of the theoretical model can be found in [GBS98]. A variational formulation using such an approach was proposed by M. Fremond [Fre82a, b]. A development of the Fremond theory is given in [RCC99] where the step-by-step numerical method with application to the microindentation problem is proposed. Application of nonlocal adhesion laws reduces the contact problem to one of integro-differential equations, including integrals with respect to spatial variables [HK03]. In these cases CPU time is very expensive and, therefore, parallel computing tools must be used [CKN$^+$99, OLS$^+$01, OBN$^+$00]. An example of an analytical solution is presented in [BKK03]. A nonlinear governing equation for the friction coefficient is proposed and the finite element implementation is given in [SO99]. A problem on the stress and strain in adhesively patched sheets by the duality method is considered in [SA97].

Notice that inverse problems in adhesion contact mechanics are solved (as a rule) within the framework of the macroscale approach. Numerical results for the solution of an inverse problem are presented in [BBM04] (see the bibliography in this paper).

Industrial applications of solutions to friction and adhesion contact problems concerning microengine and microactuator design are given in [MF98]. Related topics are considered in [LMM97, MN96] and others. Contact problems in these applications are treated using the plate-bending theory and plane elasticity theory. The parallel frictional contact algorithms for an industrial application are given in [ABLM99].

8.3 Numerical implementation of the contact interaction phenomena

Traditionally, mathematical analysis of contact problems has been concerned with existence theorems, uniqueness (or nonuniqueness) of solutions (the nonuniqueness problem is very close to the nonstability phenomenon) and the regularity of solutions, but nowadays the mathematical tools used are more refined.

The basis of the mathematical analysis for the contact problems can be found in the works of J.-L. Lions, G. Stampacchia and G. Duvaut (see [Lio69, Lio80, DL72]). For further work in this area see G. Fichera [Fic72], R. Glowinski [GLT81], C. Baiocchi and A. Capelo [BC84], P. D. Panagiotopoulos [Pan85, AP92] and many others.

The numerical implementation of the solutions to contact problems and their mathematical foundation (convergence theorems, error estimates, etc.) is considered in numerous publications, so that it is quite impossible to enumerate even the most fundamental parts in such a brief survey as this. However, we can give some of the references which we used in our research or which are the most recent. First we mention the basic monograph [GLT81] as well as the other publications on the subject, mentioned above. Moreover, very important for unilateral problems are the results presented in the books [HN88, HN96, KNGK04, NRKT89]. The mathematical programming method to the rolling friction contact problem is used in [AAS04].

In general, for numerical solution to contact problems, one can use commercial packages (ANSYS, NASTRAN, MARC, and others). For a comparative study of the effectiveness of these programs in contact problems see [BF04].

Some studies devoted to contact problems with finite displacements and strains should be mentioned. Here, the most difficult problem is to construct the appropriate grids for two contacting bodies. This problem was solved using mortar finite elements [BBHL99] or special finite or boundary elements realizing the impenetrability restriction "point-to-segment" or "segment-to-segment" (see [NRMP04, Section "Contact Mechanics"]). An adaptive mesh for taking into account the stresses concentration effect is developed in [BKT02].

It must be noted that the number of publications devoted to the boundary integral equation method for contact problems and their numerical realization has increased recently. An exhaustive presentation of this approach is given in [BA93]. Some recent results in this direction can be found in [Bre04, HO03, Lin02, Bur01]. Results for the contact problems can be found in [Gue02, ESW99, EW00, Ste86, GS93, BBHL99]. Boundary integral equations and boundary elements methods will clearly develop rapidly over the next few years.

Note, in conclusion, that the variational approach has been successfully used in medicine and biology: in the theory of biological membranes – filtration of biological liquids through semipermeable membranes [DL72], in computing the equilibrium forms of erythrocytes, in modeling the heat equilibrium of an organism with the environment and evaluation of the energy state in the organism as a whole, and in the study of dynamic growth and breath.

References

[AAS04] K. Addi, H. Antes, and G. E. Stavroulakis. On solving a rolling frictional contact problem using BEM and mathematical programming. *International Journal of Applied Mathematical Sciences*, 1:73–95, 2004.

[ABLM99] K. Ach, M. Barboteu, F. Lebon, and B. MBodji. Parallel frictional contact algorithms and industrial applications. *Computer Methods in Applied Mechanics and Engineering*, 177(3–4):169–181, 1999.

[Ada75] R. A. Adams. *Sobolev Spaces.* Number 65 in Pure and Applied Mathematics. Academic Press, New York, 1975.

[AHU58] K. J. Arrow, L. Hurwicz, and H. Uzawa. *Studies in Linear and Non-Linear Programming*, Vol. II, *Stanford Mathematical Studies in the Social Sciences.* Stanford University Press, Stanford, CA, 1958.

[AKS97] K. T. Andrews, K. L. Kuttler, and M. Shillor. On the dynamic behaviour of a thermoviscoelastic body in frictional contact with a rigid obstacle. *European Journal of Applied Mathematics*, 8(4):417–436, 1997.

[Ali02] M. H. Aliabadi. *The Boundary Element Method.* Vol. 2, *Applications in Solids and Structures.* Wiley, New York, 2002.

[AP92] H. Antes and P. D. Panagiotopoulos. *The Boundary Integral Approach to Static and Dynamic Contact Problems. Equality and Inequality Methods.* Number 108 in International Series of Numerical Mathematics. Birkhäuser, Basel, 1992.

[BA93] C. A. Brebbia and R. A. Adey, editors. *Boundary Element Method Educational Package.* WIT Press, Southampton, 1993.

[Bal77] J. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 63(4):337–403, 1977.

[Ban83a] N. V. Banichuk. *Problems and Methods of Optimal Structural Design.* Number 26 in Mathematical Concepts and Methods in Science and Engineering. Plenum Press, New York, 1983.

[Ban83b] N. V. Banichuk. *Problems and Methods of Optimal Structural Design.* Number 26 in Mathematical Concepts Methods in Science Engineering Plenum Press, New York, 1983.

[BBHL99] F. Ben Belgacem, P. Hild, and P. Laborde. Extension of the mortar finite element method to a variational inequality modeling unilateral contact. *Mathematical Models & Methods in Applied Sciences*, 9(2):287–303, 1999.

[BBM04] M. Bocciarelli, G. Bolzon, and G. Maier. Calibration of anisotropic elastic-plastic models for thin layers and foils in microtechnologies: two novel techniques. Lecture Presentation on XXI International Congress of Theoretical and Applied Mechanics, 15–21 August 2004, Warsaw, Poland, 2004.

[BC84] C. Baiocchi and A. Capelo. *Variational and Quasivariational Inequalities. Applications to Free Boundary Problems.* Wiley, New York, 1984.

[Ber83] V. L. Berdichevski. *Variational Principles in Continuum Medium Mechanics.* Nauka, Moscow, 1983. In Russian.

[BF91] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods.* Number 15 in Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 1991.

[BF04] D. Barlam and E. Feldman. Solution of contact problems in missile stage connections: NASTRAN versus MARC. In P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau, editors, *Proc. of the 4th European Congress on Computational Methods in Applied Sciences and Engineering, Jyväskylä, Finland, 24–28 July 2004*, Vol. I, Jyväskylä, 2004. University of Jyväskylä. CD ROM, www.mit.jyu.fi/eccomas2004

[BG04] P. E. Barbone and N. H. Gokhale. Elastic modulus imaging: on the uniqueness and nonuniqueness of the elastography inverse problem in two dimensions. *Inverse Problems*, 20(1):283–296, 2004.

[BKK03] F. M. Borodich, L. M. Keer, and C. S. Korach. Analytical study of fundamental nanoindentation test relations for indenters of non-ideal shapes. *Nanotechnology*, 14:803–808, 2003.

[BKT02] P. Béal, J. Koko, and R. Touzani. Mesh r-adaptation for unilateral contact problems. *International Journal of Applied Mathematics and Computer Science*, 12(1):9–16, 2002.

[BM82] M. Barquins and D. Maugis. Adhesive contact of axisymmetrical punches on an elastic half-space: the modified Hertz–Huber's stress tensor for contacting spheres. *Journal de Mécanique Theorique et Appliquées*, 1(2):331–357, 1982.

[Bre04] C. A. Brebbia, editor. *Boundary Elements XXVI (BEM 26).* Number 19 in Advances in Boundary Elements. WIT Press, Southampton, 2004.

[BS92] D. Baĭnov and P. Simeonov. *Integral Inequalities and Applications.* Number 57 in Mathematics and its Applications (East European Series). Kluwer Academic, Dordrecht, 1992.

[BS01] I. Babuška and T. Strouboulis. *The Finite Element Method and Its Reliability.* Clarendon Press/Oxford University Press, New York, 2001.

[Bur01] T. Burczynski, editor. *IUTAM/IACM/IABEM Symposium on Advanced Mathematical and Computational Mechanics Aspects of the Boundary Element Method*, Springer-Verlag, Berlin, 2001.

[BZX04] T. Belytschko, S. Zhang, and S. Xiao. Coupled molecular/continuum method. In P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau, editors, *Proc. of the 4th European Congress on Computational Methods in Applied Sciences and Engineering. Jyväskylä, Finland, 24–28 July 2004*, Vol. I, Jyväskylä, 2004. University of Jyväskylä. CD ROM, www.mit.jyu.fi/eccomas2004

[Céa64] J. Céa. Approximation variationnelle des problèmes aux limites. *Annales de l'Institut Fourier*, 14(2):345–444, 1964.

[Céa71] J. Céa. *Optimisation: Théorie et Algorithmes.* Number 2 in Méthodes Mathématiques de l'Informatique. Dunod, Paris, 1971.

[CH53] R. Courant and D. Hilbert. *Methods of Mathematical Physics.* Vol. 1. Interscience Publishers, New York, 1953.

[Che79] G. P. Cherepanov. *Mechanics of Brittle Fracture.* McGraw-Hill, New York, 1979.

[Chr71] R. M. Christensen. *Theory of Viscoelasticity: An Introduction.* Academic Press, New York, 1971.

[Cia78] Ph. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* Number 4 in Studies in Mathematics and its Applications. North-Holland, Amsterdam, 1978.

[Cia88] Ph. G. Ciarlet. *Mathematical Elasticity. Vol. I. Three-Dimensional Elasticity.* Number 20 in Studies in Mathematics and its Applications. North-Holland, Amsterdam, 1988.

[CK85] J. H. Cheng and N. Kikuchi. An incremental constitutive relation of unilateral contact friction for large deformation analysis. *Journal of Applied Mechanics*, 52(3):639–648, 1985.

[CKN+99] T. Campbell, R. K. Kalia, A. Nacano, P. Vashishta, S. Ogata, and S. Rodgers. Dynamics of oxidation of aluminium nanoclusters using variable charge molecular-dynamics simulations on parallel computers. *Physical Review Letters*, 82(24):4866–4869, 1999.

[CL91] Ph. G. Ciarlet and J.-L. Lions, editors. *Handbook of numerical analysis. Vol. II. Finite element methods.* Part 1. North-Holland, Amsterdam, 1991.

[CL96] Ph. G. Ciarlet and J.-L. Lions, editors. *Handbook of numerical analysis. Vol. IV. Finite element methods.* Part 2. *Numerical methods for solids. Part 2.* North-Holland, Amsterdam, 1996.

[CMEAR01] A. Czekanski, S. A. Meguid, N. El-Abbasi, and M. H. Refaat. On the elastodynamic solution of frictional contact problems using variational inequalities. *International Journal for Numerical Methods in Engineering*, 50(3):611–627, 2001.

[Con95] A. Constantinescu. On the identification of elastic moduli from displacement-force boundary measurements. *Inverse Problems in Engineering*, 1:293–313, 1995.

[CR80] Ph. G. Ciarlet and P. Rabier. *Les Équations de von Kármán.* Number 826 in Lectures Notes in Mathematics. Springer-Verlag, Berlin, 1980.

[CT00] A. Constantinescu and N. Tardieu. On the determination of elastic coefficients from indentation experiments. *Inverse Problems*, 16(3):577–588, 2000.

[Din98] X. P. Ding. Set-valued implicit Wiener–Hopf equations and generalized strongly nonlinear quasivariational inequalities. *Journal of Applied Analysis*, 4(1):75–87, 1998.

[DKS73] B. V. Deryagin, N. A. Krotova, and V. P. Smilga. *Adhesion of Solids.* Nauka, Moscow, 1973. In Russian.

[DL72] G. Duvaut and J.-L. Lions. *Les Inéquations en Mécanique et en Physique.* Number 21 in Travaux et Recherches Mathématiques. Dunod, Paris, 1972.

[DL90] R. Dautray and J.-L. Lions. *Mathematical Analysis and Numerical Methods for Science and Technology.* Vol. 4. *Integral Equations and Numerical Methods.* Springer-Verlag, Berlin, 1990.

[Doe74] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation.* Springer-Verlag, Berlin, 1974.

[EMT69] A. B. Efimov, V. I. Maly, and N. M. Tolkacheva. Contact problem for a coating elastic body. *Communications of AN SSSR, Mechanics of Solids*, 1:166–171, 1969. In Russian.

[Esv01] C. Esveld. *Modern Railway Track.* MRT-Productions, Zaltbommel, 2nd edition, 2001.

[ESW99] C. Eck, O. Steinbach, and W. L. Wendland. A symmetric boundary element method for contact problems with friction. *Mathematics and Computers in Simulation*, 50(1–4):43–61, 1999.

[ET74] I. Ekeland and R. Temam. *Analyse Convexe et Problèmes Variationnels.* Dunod and Gauthier-Villars, Paris, 1974.

[EW00] Ch. Eck and W. L. Wendland. An adaptive boundary element method for contact problems. In M. Bonnet, A.-M. Sändig, and W. L. Wendland, editors, *Mathematical Aspects of Boundary Element Methods (Palaiseau, 1998)*, number 414 in Research Notes in Mathematics, pages 116–127, Chapman & Hall/CRC Press, Boca Raton, FL, 2000.

[Fed78] R. P. Fedorenko. *Approximate Solution of Optimal Control Problems.* Nauka, Moscow, 1978. In Russian.

[FGVH02] J. R. Fernández-García, J. M. Viaño, and P. Hild. Numerical approximation of the elastic-viscoplastic contact problem using noncoinciding finite element meshes. In J. A. C. Martins and M. D. P. Monteiro Marques, editors, *Contact Mechanics (Praia da Consolação, 2001)*, number 103 in Solid Mechanics and Its Applications, pages 395–402, Kluwer Academic Dordrecht, 2002.

[Fic72] G. Fichera. Boundary value problems in elasticity with unilateral constraints. In S. Flügge, editor, *Encyclopedia in Physics VI a/2, Mechanics of solids II*, pages 347–389. Springer-Verlag, Berlin, 1972.

[FK80] S. Fučík and A. Kufner. *Nonlinear Differential Equations.* Number 2 in Studies in Applied Mechanics. Elsevier, Amsterdam, 1980.

[Fle81] R. Fletcher. *Practical Methods of Optimization.* Vol. 2. *Constrained Optimization.* Wiley, Chichester, 1981.

[FM68] A. V. Fiacco and G. P. McCormick. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* Wiley, New York, 1968.

[Fre82a] M. Fremond. Adherence des solides. *Comptes rendus de l'Académie des sciences Paris Série II*, 295(2):769–772, 1982.

[Fre82b] M. Fremond. Équilibre de structures qui adhèrent à leur support. *Comptes rendus de l'Académie des sciences Paris Série II*, 295:913–916, 1982.

[Fre85] M. Fremond. Frottement, adhesion, lubrification, dissipation dans l'adherence des solides. *Comptes rendus de l'Académie des sciences Paris Série II*, 300(15):709–714, 1985.

[GA60] A. E. Green and J. E. Adkins. *Large Elastic Deformations, and Non-Linear Continuum Mechanics.* Clarendon Press, Oxford, 1960.

[Gal48] L. A. Galin. On the pressure of a solid body on a plate. *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 12:345–348, 1948.

[GBS98] J. F. Ganghoffer, A. Brillard, and J. Schultz. Mechanical modelling of adhesion and adherence between two solids. *European Journal of Applied Mathematics*, 9(3):305–326, 1998.

[Ger73] P. Germain. *Cours de Mécanique des Milieux Continus. Tome I: Théorie Générale*. Masson et Cie, Paris, 1973.

[Gla80] G. M. L. Gladwell. *Contact Problems in the Classical Theory of Elasticity*. Number 1 in Monographs and Textbooks on Mechanics of Solids and Fluids: Mechanics of Elastic and Viscoelastic Solids. Martinus Nijhoff Publishers, The Hague, 1980.

[Gla04] G. M. L. Gladwell. *Inverse Problems in Vibration*. Number 119 in Solid Mechanics and its Applications. Kluwer Academic, Dordrecht, 2nd edition, 2004.

[Glo84] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*. Springer-Verlag, New York, 1984.

[GLT81] R. Glowinski, J.-L. Lions, and R. Trémolières. *Numerical Analysis of Variational inequalities*. Number 8 in Studies in Mathematics and its Applications. North-Holland, Amsterdam, 1981.

[Gor98] I. G. Goryacheva. *Contact Mechanics in Tribology*. Number 61 in Solid Mechanics and its Applications. Kluwer Academic, Dordrecht, 1998.

[GPS02] H. Goldstein, C. P. Poole, and J. L. Safko. *Classical Mechanics*. Addison-Wesley, Boston, MA, 3rd edition, 2002.

[GS93] J. Gwinner and E. P. Stephan. A boundary element procedure for contact problems in plane linear elastostatics. *RAIRO Modélisation Mathématique et Analyse Numérique*, 27(4):457–480, 1993.

[Gue02] H. Guediri. On a boundary variational inequality of the second kind modelling a friction problem. *Mathematical Methods in the Applied Sciences*, 25(2):93–114, 2002.

[Gur63] M. E. Gurtin. Variational principles in the linear theory of viscoelasticity. *Archive for Rational Mechanics and Analysis*, 13:179–191, 1963.

[Gur64a] M. E. Gurtin. Variational principles for linear elastodynamics. *Archive for Rational Mechanics and Analysis*, 16:34–50, 1964.

[Gur64b] M. E. Gurtin. Variational principles for linear initial value problems. *Quarterly Journal of Applied Mathematics*, 22(3):252–256, 1964.

[Gya70] I. Gyarmati. *Non-equilibrium Thermodynamics. Field Theory and Variational Principles*. Springer-Verlag, Berlin, 1970.

[HA79] E. J. Haug and J. S. Arora. *Applied Optimal Design: Mechanical and Structural Systems*. Wiley, Chichester, 1979.

[HC93] Q.-C. He and A. Curnier. Anisotropic dry friction between two orthotropic surfaces undergoing large displacements. *European Journal of Mechanics. A Solids*, 12(5):631–666, 1993.

[HdSM02] M. Hjiaj, G. de Saxcé, and Z. Mróz. A variational inequality-based formulation of the frictional contact law with a non-associated sliding rule. *European Journal of Mechanics. A. Solids*, 21(1):49–59, 2002.

[Hel84] K. Hellan. *Introduction to Fracture Mechanics*. McGraw-Hill, New York, 1984.

[Her95] H. Hertz. Über die Berührung fester elastischer Körper. In H. Hertz, editor, *Gesammelte Werke, Band 1*, pages 283–287. Johan Ambrosius Barth, Leipzig, 1895.

[Her65] L. R. Herrmann. Elasticity equations of incompressible and nearly incompressible materials by a variational theorem. *American Institute of Aeronautics and Astronautics Journal*, 3(10):1896–1900, 1965.

[HHNL88] I. Hlaváček, J. Haslinger, J. Nečas, and J. Lovíšek. *Numerical Solution of Variational Inequalities.* Number 66 in Springer Series in Applied Mathematical Sciences. Springer-Verlag, New York, 1988.

[Hil98] R. Hill. *The Mathematical Theory of Plasticity. Reprint of the 1950 original.* Number 11 in Oxford Engineering Science Series. The Clarendon Press Oxford University Press, New York, 1998.

[HK03] S. Hong and K.-S. Kim. Extraction of cohesive-zone laws from elastic far-fields of a cohesive crack tip: a field projection method. *Journal of the Mechanics and Physics of Solids*, 51(7):1267–1286, 2003.

[HN88] J. Haslinger and P. Neittaanmäki. *Finite Element Approximation for Optimal Shape Design. Theory and Applications.* Wiley, Chichester, 1988.

[HN96] J. Haslinger and P. Neittaanmäki. *Finite Element Approximation for Optimal Shape, Material and Topology Design.* Wiley, Chichester, 2nd edition, 1996.

[HO03] W. S. Hall and G. Oliveto, editors. *Boundary Element Methods for Soil-Structure Interaction.* Kluwer Academic, Dordrecht, 2003.

[Ign63] J. Ignaczak. A completeness problem for stress equations of motion in the linear elasticity theory. *Archivum Mechaniki Stosowanej*, 15(2):225–234, 1963.

[Ily49] A. A. Ilyushin. The theory of plasticity in the case of simple loading accompanied by strain-hardening. *Tech. Memos. Nat. Adv. Comm. Aeronaut.*, 1949(1207), 1949.

[IN02] I. R. Ionescu and Q.-L. Nguyen. Dynamic contact problems with slip-dependent friction in viscoelasticity. *International Journal of Applied Mathematics and Computer Science*, 12(1):71–80, 2002.

[IP70] A. A. Ilyushin and B. E. Pobedrya. *Foundations of the Mathematical Theory of Thermal Viscoelasticity.* Nauka, Moscow, 1970. In Russian.

[Iva89] E. B. Ivanoff. *Contact Problems with Adhesion.* Ph.D. thesis, Institute for the Problems in Mechanics RAN, Moscow, 1989. In Russian.

[JKR71] K. L. Johnson, K. Kendall, and A. D. Roberts. Surface energy and the contact of elastic solids. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 324(1558):301–313, 1971.

[Joh85] K. L. Johnson. *Contact Mechanics.* Cambridge University Press, Cambridge, 1985.

[Kal66] J. J. Kalker. Rolling with slip and spin in presence of dry friction. *Wear*, 9:20–38, 1966.

[Kan39] L. V. Kantorovich. *The Mathematical Method of Production Planning and Organization.* Gostexteorizdat, Moscow, 1939. In Russian.

[KH95] A. S. Khan and S. Huang. *Continuum Theory of Plasticity.* Wiley, New York, 1995.

[KI94] A. S. Kravchuk and H. Ivanoff. Sur la résolution d'un problème de contact avec adhésion, pour une loi non-local de dissipation visqeuse. *Comptes rendus de l'Académie des, Sciences, de Paris, Serie 2*, 318(5):567–570, 1994.

[KMU85] A. S. Kravchuk, V. P. Maĭboroda, and Yu. S. Urzhumtsev. *Mechanics of Polymer and Composite Materials*. Nauka, Moscow, 1985. In Russian.

[KN03a] A. S. Kravchuk and P. Neittaanmäki. Dynamic identification of the deformed body parameters. In G. C. Cohen, E. Heikkola, P. Joly, and P. Neittaanmäki, editors, *Mathematical and Numerical Aspects of Wave Propagation – WAVES 2003*, pages 577–581, Springer-Verlag, Berlin, 2003.

[KN03b] A. S. Kravchuk and P. Neittaanmäki. Variational method for the dynamic contact problems of the elastic bodies. In V. I. Belokon, editor, *Mathematical Problems of the Theory of Elasticity (Rostov-na-Donu, 2003)*, pages 77–82, Rostov University Press, Rostov-na-Donu, 2003.

[KNG04] A. S. Kravchuk, P. Neittaanmäki, and I. G. Goryacheva. Boundary value problems and inequalities in contact mechanics. In P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau, editors, *Proc. of the 4th European Congress on Computational Methods in Applied Sciences and Engineering, Jyväskylä, Finland, 24–28 July 2004*, volume I, Jyväskylä, 2004. University of Jyväskylä. CD ROM, www.mit.jyu.fi/eccomas2004

[KNGK04] M. Křížek, P. Neittaanmäki, R. Glowinski, and S. Korotov, editors. *Conjugate Gradient Algorithms and Finite Element Methods*. Springer-Verlag, Berlin, 2004

[KO88] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*. Number 8 in SIAM Studies in Applied Mathematics. SIAM, Philadelphia, PA, 1988.

[Kor07] A. Korn. Sur les équations de l'élasticité. *Annales Scientifiques de l'École Normale Supérieure. Troisième Série*, 24(3):9–75, 1907.

[Kra78] A. S. Kravchuk. Formulation of the problem of contact between several deformable bodies as a nonlinear programming problem. *Journal of Applied Mathematics and Mechanics*, 42(3):466–474, 1978.

[Kra80] A. S. Kravchuk. On the theory of contact problems taking account of friction on the contact surface. *Journal of Applied Mathematics and Mechanics*, 44(1):122–129, 1980.

[Kra97] A. S. Kravchuk. *Variational and Quasivariational Inequalities in Mechanics*. MGAPI, Moscow, 1997. In Russian.

[Kra01] A. S. Kravchuk. Application of Ilyushin's definitions for simple and complex loading in the contact problem with friction. In I. A. Kiyko and G. A. Brovko, editors, *Proc. of Int. Sci. Symposium on the Deformed Solids Problems dedicated to 90th Anniversary of A. A. Ilyusin, Elasticity and Non-Elasticity*, pages 182–187. Moscow University Press, Moscow, 2001.

[Kre71] S. G. Kreĭn. *Linear Differential Equations in Banach Space*. Number 29 in Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1971.

[KS80] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and their Applications*. Number 88 in Pure and Applied Mathematics. Academic Press, New York, 1980.

[KS81] A. S. Kravchuk and V. A. Sursyakov. Numerical solution of geometrically nonlinear contact problems. *Soviet Physics, Doklady* 26:789–790, 1981.

[KS04] K. Kuttler and M. Shillor. Dynamic contact with Signorini condition and slip rate dependent friction. *Electronic Journal of Differential Equations*, 2004(83):21 p., 2004.

[KT51] H. W. Kuhn and A. W. Tucker. Nonlinear programming. In *Proc. Second Berkeley Symp. on Math. Statistics and Probability*, pages 481–492, Berkeley University Press, Berkeley, 1951.

[Leo50] M. A. Leontovich. *An introduction to Thermodynamics.* GITTL Publishers, Moscow, 1950. In Russian.

[Lin02] A. M. Linkov. *Boundary Integral Equations in Elasticity Theory.* Number 99 in Solid Mechanics and its Applications. Kluwer Academic, Dordrecht, 2002.

[Lio68] J.-L. Lions. *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles.* Dunod and Gauthier-Villars, Paris, 1968.

[Lio69] J.-L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* Dunod and Gauthier-Villars, Paris, 1969.

[Lio75] J.-L. Lions. On free surface problems: Methods of variational and quasi variational inequalities. In *Computational Mechanics (Internat. Conf. on Computational Methods in Nonlinear Mech., Austin, TX, 1974)*, number 461 in Lecture Notes in Math., pages 129–148, Springer-Verlag, Berlin, 1975.

[Lio80] J.-L. Lions. The work of G. Stampacchia in variational inequalities. In *Variational Inequalities and Complementarity Problems (Proc. Internat. School, Erice, 1978)*, pages 1–24, Wiley, Chichester, 1980.

[LL50] M. A. Lavrentyev and L. A. Lusternik. *Course of the Calculus of Variation.* Gostekhteorizdat, Moscow, 1950. In Russian.

[LM73] J.-L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications.* Vol. III. Springer-Verlag, New York, 1973.

[LMM97] P. Le Moal and P. Minotti. A 2-D analytical approach of the stator-rotor contact problem including rotor bending effects for high torque piezomotor design. *European Journal of Mechanics. A Solids*, 16(6):1067–1103, 1997.

[Lov44] A. E. H. Love. *A Treatise on the Mathematical Theory of Elasticity.* Dover Publications, New York, 4th edition, 1944.

[LVG02] L. P. Lebedev, I. I. Vorovich, and G. M. L Gladwell. *Functional Analysis. Applications in Mechanics and Inverse Problems.* Number 100 in Solid Mechanics and its Applications. Kluwer Academic, Dordrecht, 2002.

[MB78] D. Maugis and M. Barquins. Fracture mechanics and the adherence of viscoelastic bodies. *Journal of Physics D: Applied Physics*, 11(14):1989–2023, 1978.

[MB87] J. Mikusiński and T. K. Boehme. *Operational Calculus.* Vol. II. Number 110 in International Series of Monographs in Pure and Applied Mathematics. Pergamon Press, Oxford, 1987.

[McC57] A. J. McConnell. *Application of Tensor Analysis.* Dover Publications, New York, 1957.

[MF53] Ph. M. Morse and H. Feshbach. *Methods of Theoretical Physics.* McGraw-Hill, New York, 1953.

[MF98] P. Minotti and A. Ferreira. *Les Micromachines.* Hermes, Paris, 1998.

[Mik64a] S. G. Mikhlin. *Integral Equations and their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology.* Macmillan, New York, 2nd edition, 1964.

[Mik64b] S. G. Mikhlin. *Variational Methods in Mathematical Physics.* Macmillan, New York, 1964.

[Mik65] S. G. Mikhlin. *Multidimensional Singular Integrals and Integral Equations.* Pergamon Press, Oxford, 1965.

[MM78] Z. Mroz and R. Michalowski. Associated and nonassociated sliding rules in contact friction problems. *Archives of Mechanics (Archiwum Mechaniki Stosowanej)*, 30(3):259–276, 1978.

[MM02] J. A. C. Martins and M. D. P. Monteiro Marques, editors. *Contact Mechanics: Proc. of the 3rd Contact Mechanics International Symposium (CMIS 2001) (Praia da Consolação, 2001)*, number 103 in Solid Mechanics and its Applications, Kluwer Academic, Dordrecht, 2002.

[MN96] P. Minotti and V. Nocton. Etude théorique d'un microtranslateur piézoélectrique à ondes stationnaires: Une introduction aux futurs micromoteurs multi degrés de liberté. *Mechanism and Machine Theory*, 31(1):91–108, 1996.

[Mor84] N. F. Morozov. *Mathematical Problems of Fracture Theory.* Nauka, Moscow, 1984. In Russian.

[Mus53] N. I. Muskhelishvili. *Some Basic Problems of Mathematical Theory of Elasticity.* Noordhoff, Groningen, 1953.

[MY04] J. R. McLaughlin and J.-R. Yoon. Unique identifiability of elastic parameters from time-dependent interior displacement measurement. *Inverse Problems*, 20(1):25–45, 2004.

[Neč67] J. Nečas. *Les Méthodes Directes en Théorie des Équations Elliptiques.* Masson, Paris, 1967.

[NH80] J. Nečas and I. Hlaváček. *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction.* Number 3 in Studies in Applied Mathematics. Elsevier, Amsterdam, 1980.

[Now70] W. Nowacki. *Teoria Sprężystości.* PWN, Warszawa, 1970.

[Now75] W. Nowacki. *Dynamic Problems of Thermoelasticity.* Noordhoff, Leyden, 1975.

[NP95] Z. Naniewicz and P. D. Panagiotopoulos. *Mathematical Theory of Hemivariational Inequalities and Applications.* Number 188 in Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, New York, 1995.

[NR04] P. Neittaanmäki and S. Repin. *Reliable Methods for Computer Simulation. Error Control and A Posteriori Estimates.* Number 33 in Studies in Mathematics and its Applications. Elsevier, Amsterdam, 2004.

[NRKT89] G. L. Nemhauser, A. H. G. Rinnooy Kan, and M. J. Todd, editors. *Optimization.* Number 1 in Handbooks in Operations Research and Management Science. North-Holland, Amsterdam, 1989.

[NRMP04] P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau, editors. *Proc. of the 4th European Congress on Computational Methods in Applied Sciences and Engineering. Jyväskylä, Finland, 24–28 July 2004.* Vol. I. University of Jyväskylä, Jyväskylä, 2004. CD ROM, www.mit.jyu.fi/eccomas2004

[NS60] V. V. Nemytskii and V. V. Stepanov. *Qualitative Theory of Differential Equations*. Number 22 in Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1960.

[NST06] P. Neittaanmäki, J. Sprekels, and D. Tiba. *Optimization of Elliptic Systems. Theory and Applications*. Springer-Verlag, Berlin, 2006.

[OBN$^+$00] A. Omeltchenko, M. E. Bachlechner, A. Nakano, R. K. Kalia, P. Vashishta, I. Ebbsjö, A. Madhukar, and P. Messina. Stress domains in $si(111)/a - si_3n_4$ nanopixel: Ten-million-atom molecular dynamics simulations on parallel computers. *Physical Review Letters*, 84(2):318–321, 2000.

[Ogd84] R. W. Ogden. *Nonlinear Elastic Deformations*. Ellis Horwood, Chichester, 1984.

[OLS$^+$01] S. Ogata, E. Lidorikis, F. Shimojo, A. Nakano, P. Vashishta, and R. K. Kalia. Hybrid finite-element/molecular-dynamics/electronic-density-functional approach to materials simulations on parallel computers. *Computer Physics Communications*, 138(2):143–154, 2001.

[Ost61] M. V. Ostrogradskiĭ. *Collected works*. Vol. I–III. Izdat. Akad. Nauk Ukrain, Kiev, 1959, 1961, 1961. In Russian.

[Pan85] P. D. Panagiotopoulos. *Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions*. Birkhäuser, Boston, MA, 1985.

[PBGM64] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Macmillan, New York, 1964.

[Ped89] G. K. Pedersen. *Analysis Now*. Number 118 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.

[Pet95] I. Petersson. *Optimization of Structures in Unilateral Contact*. Ph.D. thesis, Linköping University, 1995.

[PG05] B. E. Pobedrya and D. V. Georgievskiĭ. *Foundations of the Mechanics of Continuous Medium*. Fizmathlit, Moscow, 2005. In Russian.

[PKM$^+$02] A. Pandolfi, C. Kane, J. Marsden, E., and M. Ortiz. Time-discretized variational formulation of non-smooth frictional contact. *International Journal for Numerical Methods in Engineering*, 53(8):1801–1829, 2002.

[Pob84] B. E. Pobedrya. *Mechanics of Composite Materials*. Moscow State University Press, Moscow, 1984. In Russian.

[Pra74] W. Prager. *Principles of Optimal Design: Modeling and Computation*. Cambridge University Press, Cambridge, 1974.

[Rab69] Yu. N. Rabotnov. *Creep Problems in Structural Members*. North-Holland, Amsterdam, 1969.

[RCC99] M. Raous, C. Cangémi, and M. Cocu. A consistent model coupling adhesion, friction, and unilateral contact. *Computer Methods in Applied Mechanics and Engineering*, 177(3–4):383–399, 1999.

[Rek77] K. Rektorys. *Variational Methods of Mathematics, Science and Engineering*. D. Reidel Publishing, Dordrecht, 1977.

[Rep89] S. I. Repin. A variation-difference method for solving problems with functionals of linear growth. *U.S.S.R. Computational Mathematics and Mathematical Physics*, 29(3):35–46, 1989.

[RKMO02] A. Randolfi, C. Kane, J. E. Marsden, and M. Ortiz. Time discretized variational formulation of non-smooth frictional contact. *Journal of Physics Condened Matter*, 53:1801–1829, 2002.

[Roc70] R. T. Rockafellar. *Convex Analysis.* Number 28 in Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970.

[RP01] J.-M. Ricaud and E. Pratt. Analysis of a time discretization for an implicit variational inequality modelling dynamic contact problems with friction. *Mathematical Methods in the Applied Sciences*, 24(7):491–511, 2001.

[RTHC98] H. Rafii-Tabar, L. Hua, and M. Cross. A multi-scale atomistic-continuum modeling of crack propagation in a two-dimensional macroscopic plate. *Journal of Physics: Condensed Matter*, 10(11):2375–2387, 1998.

[SA97] N. K. Salgado and M. H. Aliabadi. A dual reciprocity method for the analysis of adhesively patched sheets. *Communications in Numerical Methods in Engineering*, 13(5):397–405, 1997.

[Sed97] L. I. Sedov. *Mechanics of Continuous Media.* Vols. 1, 2. Number 4 in Series in Theoretical and Applied Mechanics. World Scientific Publishing River Edge, NJ, 1997.

[Sig33] A. Signorini. Sopra alcune questioni di elastostatica. *Atti Soc. Ital. per il Progresso delle Scienze*, 1933.

[Sig59] A. Signorini. Questioni di elasticità non linearizzata e semilinearizzata. *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, 18:95–139, 1959.

[Sle02] L. I. Slepyan. *Models and Phenomena in Fracture Mechanics.* Springer-Verlag, Berlin, 2002.

[SO99] U. Sellgren and U. Olofsson. Application of a constitutive model for micro-slip in finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 170(1–2):65–77, 1999.

[Sob50] S. L. Sobolev. *Some Applications of Functional Analysis in Mathematical Physics.* Leningrad State University, Leningrad, 1950. In Russian.

[Sok64] I. S. Sokolnikoff. *Tensor Analysis. Theory and Applications to Geometry and Mechanics of Continua.* Wiley, New York, 2nd edition, 1964.

[Spe75] D. A. Spence. The Hertz contact problem with finite friction. *Journal of Elasticity*, 5(3–4):297–319, 1975.

[Spe77] A. A. Spector. A variational method of investigation of contact problems involving slippage and adhesion. *Doklady Akademii Nauk SSSR*, 236(1):39–42, 1977.

[Ste86] E. Stephan. A boundary integral equation method for three-dimensional crack problems in elasticity. *Mathematical Methods in the Applied Sciences*, 8(4):609–623, 1986.

[Ste02] E. Stein, editor. *Error-Controlled Adaptive Finite Elements in Solid Mechanics.* Wiley, Chichester, 2002.

[TA79] A. N. Tikhonov and V. Ya. Arsenin. *Methods for the Solution of Ill-Posed Problems.* Nauka, Moscow, 2nd edition, 1979. In Russian.

[Tar74] L. Tartar. Inéquations quasi variationnelles abstraites. *Comptes rendus de l'Académie des sciences Paris Série A*, 278:1193–1196, 1974.

[Tem86] R. Temam. A generalized Norton–Hoff model and the Prandtl–Reuss law of plasticity. *Archive for Rational Mechanics and Analysis*, 95(2):137–183, 1986.

[Tik86] V. M. Tikhomirov. *Fundamental Principles of the Theory of Extremal Problems.* Wiley, Chichester, 1986.

[Tim87] S. P. Timoshenko. *Theory of Elasticity.* McGraw-Hill, New York, 3rd edition, 1987.

[Tit48] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals.* Clarendon Press, Oxford, 3rd edition, 1948.

[TP69] P. Tong and T. H. H. Pian. A variational principle and the convergence of a finite-element method based on assumed stress distribution. *International Journal of Solids and Structures*, 5(5):463–472, 1969.

[Vaĭ64] M. M. Vaĭnberg. *Variational Methods for the Study of Nonlinear Operators.* Holden-Day Series in Mathematical Physics. Holden-Day, San Francisco, CA, 1964.

[VH85] A. I. Volpert and S. I. Hudyaev. *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics.* Martinus Nijhoff Publishers, Dordrecht, 1985.

[VS81] A. V. Vovkushevsky and B. A. Shoikhet. *Calculus of Large Scale Hydrotechnical Structures Taking into Account the Seam Opening.* Energoizdat, Moscow, 1981. In Russian.

[Was68] K. Washizu. *Variational Methods in Elasticity and Plasticity.* Number 9 in International Series of Monographs in Aeronautics and Astronautics. Pergamon Press, Oxford, 1968.

[Yak90] V. G. Yakhno. *Inverse Problems for Differential Equations of Elasticity.* Nauka, Novosibirsk, 1990. In Russian.

[Yos65] K. Yosida. *Functional Analysis.* Springer-Verlag, Berlin, 1965.

[You69] L. C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory.* W. B. Saunders, Philadelphia, PA, 1969.

[Zie63] H. Ziegler. Some extremum principles in irreversible thermodynamics with application to continuum mechanics. In *Progress in Solid Mechanics.* Vol. IV, pages 91–193. North-Holland, Amsterdam, 1963.

[Zie83] H. Ziegler. *An Introduction to Thermomechanics.* Number 21 in North-Holland Series in Applied Mathematics and Mechanics. North-Holland, Amsterdam, 2nd edition, 1983.

[ZT00] O. C. Zienkiewicz and R. L. Taylor. *The Finite Element Method.* Vol. 1–3. Butterworth-Heinemann, Oxford, 5th edition, 2000.

Index

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

Aims and Scope of the Series

The fundamental questions arising in mechanics are: *Why?*, *How?*, and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids. The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies; vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

1. R.T. Haftka, Z. Gürdal and M.P. Kamat: *Elements of Structural Optimization*. 2nd rev.ed., 1990 ISBN 0-7923-0608-2
2. J.J. Kalker: *Three-Dimensional Elastic Bodies in Rolling Contact.* 1990 ISBN 0-7923-0712-7
3. P. Karasudhi: *Foundations of Solid Mechanics.* 1991 ISBN 0-7923-0772-0
4. *Not published*
5. *Not published.*
6. J.F. Doyle: *Static and Dynamic Analysis of Structures.* With an Emphasis on Mechanics and Computer Matrix Methods. 1991 ISBN 0-7923-1124-8; Pb 0-7923-1208-2
7. O.O. Ochoa and J.N. Reddy: *Finite Element Analysis of Composite Laminates.* ISBN 0-7923-1125-6
8. M.H. Aliabadi and D.P. Rooke: *Numerical Fracture Mechanics.* ISBN 0-7923-1175-2
9. J. Angeles and C.S. López-Cajún: *Optimization of Cam Mechanisms.* 1991 ISBN 0-7923-1355-0
10. D.E. Grierson, A. Franchi and P. Riva (eds.): *Progress in Structural Engineering.* 1991 ISBN 0-7923-1396-8
11. R.T. Haftka and Z. Gürdal: *Elements of Structural Optimization.* 3rd rev. and exp. ed. 1992 ISBN 0-7923-1504-9; Pb 0-7923-1505-7
12. J.R. Barber: *Elasticity*. 1992 ISBN 0-7923-1609-6; Pb 0-7923-1610-X
13. H.S. Tzou and G.L. Anderson (eds.): *Intelligent Structural Systems.* 1992 ISBN 0-7923-1920-6
14. E.E. Gdoutos: *Fracture Mechanics.* An Introduction. 1993 ISBN 0-7923-1932-X
15. J.P. Ward: *Solid Mechanics*. An Introduction. 1992 ISBN 0-7923-1949-4
16. M. Farshad: *Design and Analysis of Shell Structures*. 1992 ISBN 0-7923-1950-8
17. H.S. Tzou and T. Fukuda (eds.): *Precision Sensors, Actuators and Systems*. 1992 ISBN 0-7923-2015-8
18. J.R. Vinson: *The Behavior of Shells Composed of Isotropic and Composite Materials*. 1993 ISBN 0-7923-2113-8
19. H.S. Tzou: *Piezoelectric Shells*. Distributed Sensing and Control of Continua. 1993 ISBN 0-7923-2186-3
20. W. Schiehlen (ed.): *Advanced Multibody System Dynamics*. Simulation and Software Tools. 1993 ISBN 0-7923-2192-8
21. C.-W. Lee: *Vibration Analysis of Rotors.* 1993 ISBN 0-7923-2300-9
22. D.R. Smith: *An Introduction to Continuum Mechanics.* 1993 ISBN 0-7923-2454-4
23. G.M.L. Gladwell: *Inverse Problems in Scattering*. An Introduction. 1993 ISBN 0-7923-2478-1

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

24. G. Prathap: *The Finite Element Method in Structural Mechanics*. 1993 ISBN 0-7923-2492-7
25. J. Herskovits (ed.): *Advances in Structural Optimization.* 1995 ISBN 0-7923-2510-9
26. M.A. González-Palacios and J. Angeles: *Cam Synthesis.* 1993 ISBN 0-7923-2536-2
27. W.S. Hall: *The Boundary Element Method.* 1993 ISBN 0-7923-2580-X
28. J. Angeles, G. Hommel and P. Kovács (eds.): *Computational Kinematics*. 1993 ISBN 0-7923-2585-0
29. A. Curnier: *Computational Methods in Solid Mechanics*. 1994 ISBN 0-7923-2761-6
30. D.A. Hills and D. Nowell: *Mechanics of Fretting Fatigue*. 1994 ISBN 0-7923-2866-3
31. B. Tabarrok and F.P.J. Rimrott: *Variational Methods and Complementary Formulations in Dynamics.* 1994 ISBN 0-7923-2923-6
32. E.H. Dowell (ed.), E.F. Crawley, H.C. Curtiss Jr., D.A. Peters, R. H. Scanlan and F. Sisto: *A Modern Course in Aeroelasticity*. Third Revised and Enlarged Edition. 1995 ISBN 0-7923-2788-8; Pb: 0-7923-2789-6
33. A. Preumont: *Random Vibration and Spectral Analysis.* 1994 ISBN 0-7923-3036-6
34. J.N. Reddy (ed.): *Mechanics of Composite Materials*. Selected works of Nicholas J. Pagano. 1994 ISBN 0-7923-3041-2
35. A.P.S. Selvadurai (ed.): *Mechanics of Poroelastic Media*. 1996 ISBN 0-7923-3329-2
36. Z. Mróz, D. Weichert, S. Dorosz (eds.): *Inelastic Behaviour of Structures under Variable Loads*. 1995 ISBN 0-7923-3397-7
37. R. Pyrz (ed.): *IUTAM Symposium on Microstructure-Property Interactions in Composite Materials.* Proceedings of the IUTAM Symposium held in Aalborg, Denmark. 1995 ISBN 0-7923-3427-2
38. M.I. Friswell and J.E. Mottershead: *Finite Element Model Updating in Structural Dynamics*. 1995 ISBN 0-7923-3431-0
39. D.F. Parker and A.H. England (eds.): *IUTAM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solid Mechanics.* Proceedings of the IUTAM Symposium held in Nottingham, U.K. 1995 ISBN 0-7923-3594-5
40. J.-P. Merlet and B. Ravani (eds.): *Computational Kinematics '95*. 1995 ISBN 0-7923-3673-9
41. L.P. Lebedev, I.I. Vorovich and G.M.L. Gladwell: *Functional Analysis.* Applications in Mechanics and Inverse Problems. 1996 ISBN 0-7923-3849-9
42. J. Menčik: *Mechanics of Components with Treated or Coated Surfaces.* 1996 ISBN 0-7923-3700-X
43. D. Bestle and W. Schiehlen (eds.): *IUTAM Symposium on Optimization of Mechanical Systems.* Proceedings of the IUTAM Symposium held in Stuttgart, Germany. 1996 ISBN 0-7923-3830-8
44. D.A. Hills, P.A. Kelly, D.N. Dai and A.M. Korsunsky: *Solution of Crack Problems*. The Distributed Dislocation Technique. 1996 ISBN 0-7923-3848-0
45. V.A. Squire, R.J. Hosking, A.D. Kerr and P.J. Langhorne: *Moving Loads on Ice Plates*. 1996 ISBN 0-7923-3953-3
46. A. Pineau and A. Zaoui (eds.): *IUTAM Symposium on Micromechanics of Plasticity and Damage of Multiphase Materials*. Proceedings of the IUTAM Symposium held in Sèvres, Paris, France. 1996 ISBN 0-7923-4188-0
47. A. Naess and S. Krenk (eds.): *IUTAM Symposium on Advances in Nonlinear Stochastic Mechanics*. Proceedings of the IUTAM Symposium held in Trondheim, Norway. 1996 ISBN 0-7923-4193-7
48. D. Ieşan and A. Scalia: *Thermoelastic Deformations.* 1996 ISBN 0-7923-4230-5

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

49. J.R. Willis (ed.): *IUTAM Symposium on Nonlinear Analysis of Fracture*. Proceedings of the IUTAM Symposium held in Cambridge, U.K. 1997 ISBN 0-7923-4378-6
50. A. Preumont: *Vibration Control of Active Structures*. An Introduction. 1997 ISBN 0-7923-4392-1
51. G.P. Cherepanov: *Methods of Fracture Mechanics: Solid Matter Physics*. 1997 ISBN 0-7923-4408-1
52. D.H. van Campen (ed.): *IUTAM Symposium on Interaction between Dynamics and Control in Advanced Mechanical Systems.* Proceedings of the IUTAM Symposium held in Eindhoven, The Netherlands. 1997 ISBN 0-7923-4429-4
53. N.A. Fleck and A.C.F. Cocks (eds.): *IUTAM Symposium on Mechanics of Granular and Porous Materials.* Proceedings of the IUTAM Symposium held in Cambridge, U.K. 1997 ISBN 0-7923-4553-3
54. J. Roorda and N.K. Srivastava (eds.): *Trends in Structural Mechanics*. Theory, Practice, Education. 1997 ISBN 0-7923-4603-3
55. Yu.A. Mitropolskii and N. Van Dao: *Applied Asymptotic Methods in Nonlinear Oscillations*. 1997 ISBN 0-7923-4605-X
56. C. Guedes Soares (ed.): *Probabilistic Methods for Structural Design.* 1997 ISBN 0-7923-4670-X
57. D. François, A. Pineau and A. Zaoui: *Mechanical Behaviour of Materials.* Volume I: Elasticity and Plasticity. 1998 ISBN 0-7923-4894-X
58. D. François, A. Pineau and A. Zaoui: *Mechanical Behaviour of Materials.* Volume II: Viscoplasticity, Damage, Fracture and Contact Mechanics. 1998 ISBN 0-7923-4895-8
59. L.T. Tenek and J. Argyris: *Finite Element Analysis for Composite Structures.* 1998 ISBN 0-7923-4899-0
60. Y.A. Bahei-El-Din and G.J. Dvorak (eds.): *IUTAM Symposium on Transformation Problems in Composite and Active Materials*. Proceedings of the IUTAM Symposium held in Cairo, Egypt. 1998 ISBN 0-7923-5122-3
61. I.G. Goryacheva: *Contact Mechanics in Tribology*. 1998 ISBN 0-7923-5257-2
62. O.T. Bruhns and E. Stein (eds.): *IUTAM Symposium on Micro- and Macrostructural Aspects of Thermoplasticity*. Proceedings of the IUTAM Symposium held in Bochum, Germany. 1999 ISBN 0-7923-5265-3
63. F.C. Moon: *IUTAM Symposium on New Applications of Nonlinear and Chaotic Dynamics in Mechanics*. Proceedings of the IUTAM Symposium held in Ithaca, NY, USA. 1998 ISBN 0-7923-5276-9
64. R. Wang: *IUTAM Symposium on Rheology of Bodies with Defects*. Proceedings of the IUTAM Symposium held in Beijing, China. 1999 ISBN 0-7923-5297-1
65. Yu.I. Dimitrienko: *Thermomechanics of Composites under High Temperatures*. 1999 ISBN 0-7923-4899-0
66. P. Argoul, M. Frémond and Q.S. Nguyen (eds.): *IUTAM Symposium on Variations of Domains and Free-Boundary Problems in Solid Mechanics*. Proceedings of the IUTAM Symposium held in Paris, France. 1999 ISBN 0-7923-5450-8
67. F.J. Fahy and W.G. Price (eds.): *IUTAM Symposium on Statistical Energy Analysis*. Proceedings of the IUTAM Symposium held in Southampton, U.K. 1999 ISBN 0-7923-5457-5
68. H.A. Mang and F.G. Rammerstorfer (eds.): *IUTAM Symposium on Discretization Methods in Structural Mechanics.* Proceedings of the IUTAM Symposium held in Vienna, Austria. 1999 ISBN 0-7923-5591-1

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

69. P. Pedersen and M.P. Bendsøe (eds.): *IUTAM Symposium on Synthesis in Bio Solid Mechanics*. Proceedings of the IUTAM Symposium held in Copenhagen, Denmark. 1999 ISBN 0-7923-5615-2
70. S.K. Agrawal and B.C. Fabien: *Optimization of Dynamic Systems*. 1999 ISBN 0-7923-5681-0
71. A. Carpinteri: *Nonlinear Crack Models for Nonmetallic Materials*. 1999 ISBN 0-7923-5750-7
72. F. Pfeifer (ed.): *IUTAM Symposium on Unilateral Multibody Contacts*. Proceedings of the IUTAM Symposium held in Munich, Germany. 1999 ISBN 0-7923-6030-3
73. E. Lavendelis and M. Zakrzhevsky (eds.): *IUTAM/IFToMM Symposium on Synthesis of Nonlinear Dynamical Systems*. Proceedings of the IUTAM/IFToMM Symposium held in Riga, Latvia. 2000 ISBN 0-7923-6106-7
74. J.-P. Merlet: *Parallel Robots*. 2000 ISBN 0-7923-6308-6
75. J.T. Pindera: *Techniques of Tomographic Isodyne Stress Analysis*. 2000 ISBN 0-7923-6388-4
76. G.A. Maugin, R. Drouot and F. Sidoroff (eds.): *Continuum Thermomechanics*. The Art and Science of Modelling Material Behaviour. 2000 ISBN 0-7923-6407-4
77. N. Van Dao and E.J. Kreuzer (eds.): *IUTAM Symposium on Recent Developments in Non-linear Oscillations of Mechanical Systems*. 2000 ISBN 0-7923-6470-8
78. S.D. Akbarov and A.N. Guz: *Mechanics of Curved Composites*. 2000 ISBN 0-7923-6477-5
79. M.B. Rubin: *Cosserat Theories: Shells, Rods and Points*. 2000 ISBN 0-7923-6489-9
80. S. Pellegrino and S.D. Guest (eds.): *IUTAM-IASS Symposium on Deployable Structures: Theory and Applications*. Proceedings of the IUTAM-IASS Symposium held in Cambridge, U.K., 6–9 September 1998. 2000 ISBN 0-7923-6516-X
81. A.D. Rosato and D.L. Blackmore (eds.): *IUTAM Symposium on Segregation in Granular Flows*. Proceedings of the IUTAM Symposium held in Cape May, NJ, U.S.A., June 5–10, 1999. 2000 ISBN 0-7923-6547-X
82. A. Lagarde (ed.): *IUTAM Symposium on Advanced Optical Methods and Applications in Solid Mechanics*. Proceedings of the IUTAM Symposium held in Futuroscope, Poitiers, France, August 31–September 4, 1998. 2000 ISBN 0-7923-6604-2
83. D. Weichert and G. Maier (eds.): *Inelastic Analysis of Structures under Variable Loads*. Theory and Engineering Applications. 2000 ISBN 0-7923-6645-X
84. T.-J. Chuang and J.W. Rudnicki (eds.): *Multiscale Deformation and Fracture in Materials and Structures*. The James R. Rice 60th Anniversary Volume. 2001 ISBN 0-7923-6718-9
85. S. Narayanan and R.N. Iyengar (eds.): *IUTAM Symposium on Nonlinearity and Stochastic Structural Dynamics*. Proceedings of the IUTAM Symposium held in Madras, Chennai, India, 4–8 January 1999 ISBN 0-7923-6733-2
86. S. Murakami and N. Ohno (eds.): *IUTAM Symposium on Creep in Structures*. Proceedings of the IUTAM Symposium held in Nagoya, Japan, 3-7 April 2000. 2001 ISBN 0-7923-6737-5
87. W. Ehlers (ed.): *IUTAM Symposium on Theoretical and Numerical Methods in Continuum Mechanics of Porous Materials*. Proceedings of the IUTAM Symposium held at the University of Stuttgart, Germany, September 5-10, 1999. 2001 ISBN 0-7923-6766-9
88. D. Durban, D. Givoli and J.G. Simmonds (eds.): *Advances in the Mechanis of Plates and Shells The Avinoam Libai Anniversary Volume*. 2001 ISBN 0-7923-6785-5
89. U. Gabbert and H.-S. Tzou (eds.): *IUTAM Symposium on Smart Structures and Structonic Systems*. Proceedings of the IUTAM Symposium held in Magdeburg, Germany, 26–29 September 2000. 2001 ISBN 0-7923-6968-8

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

90. Y. Ivanov, V. Cheshkov and M. Natova: *Polymer Composite Materials – Interface Phenomena & Processes*. 2001 ISBN 0-7923-7008-2
91. R.C. McPhedran, L.C. Botten and N.A. Nicorovici (eds.): *IUTAM Symposium on Mechanical and Electromagnetic Waves in Structured Media*. Proceedings of the IUTAM Symposium held in Sydney, NSW, Australia, 18-22 Januari 1999. 2001 ISBN 0-7923-7038-4
92. D.A. Sotiropoulos (ed.): *IUTAM Symposium on Mechanical Waves for Composite Structures Characterization*. Proceedings of the IUTAM Symposium held in Chania, Crete, Greece, June 14-17, 2000. 2001 ISBN 0-7923-7164-X
93. V.M. Alexandrov and D.A. Pozharskii: *Three-Dimensional Contact Problems*. 2001 ISBN 0-7923-7165-8
94. J.P. Dempsey and H.H. Shen (eds.): *IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics*. Proceedings of the IUTAM Symposium held in Fairbanks, Alaska, U.S.A., 13-16 June 2000. 2001 ISBN 1-4020-0171-1
95. U. Kirsch: *Design-Oriented Analysis of Structures*. A Unified Approach. 2002 ISBN 1-4020-0443-5
96. A. Preumont: *Vibration Control of Active Structures*. An Introduction (2^{nd} Edition). 2002 ISBN 1-4020-0496-6
97. B.L. Karihaloo (ed.): *IUTAM Symposium on Analytical and Computational Fracture Mechanics of Non-Homogeneous Materials.* Proceedings of the IUTAM Symposium held in Cardiff, U.K., 18-22 June 2001. 2002 ISBN 1-4020-0510-5
98. S.M. Han and H. Benaroya: *Nonlinear and Stochastic Dynamics of Compliant Offshore Structures*. 2002 ISBN 1-4020-0573-3
99. A.M. Linkov: *Boundary Integral Equations in Elasticity Theory*. 2002 ISBN 1-4020-0574-1
100. L.P. Lebedev, I.I. Vorovich and G.M.L. Gladwell: *Functional Analysis.* Applications in Mechanics and Inverse Problems (2^{nd} Edition). 2002 ISBN 1-4020-0667-5; Pb: 1-4020-0756-6
101. Q.P. Sun (ed.): *IUTAM Symposium on Mechanics of Martensitic Phase Transformation in Solids*. Proceedings of the IUTAM Symposium held in Hong Kong, China, 11-15 June 2001. 2002 ISBN 1-4020-0741-8
102. M.L. Munjal (ed.): *IUTAM Symposium on Designing for Quietness*. Proceedings of the IUTAM Symposium held in Bangkok, India, 12-14 December 2000. 2002 ISBN 1-4020-0765-5
103. J.A.C. Martins and M.D.P. Monteiro Marques (eds.): *Contact Mechanics*. Proceedings of the 3^{rd} Contact Mechanics International Symposium, Praia da Consolação, Peniche, Portugal, 17-21 June 2001. 2002 ISBN 1-4020-0811-2
104. H.R. Drew and S. Pellegrino (eds.): *New Approaches to Structural Mechanics, Shells and Biological Structures*. 2002 ISBN 1-4020-0862-7
105. J.R. Vinson and R.L. Sierakowski: *The Behavior of Structures Composed of Composite Materials*. Second Edition. 2002 ISBN 1-4020-0904-6
106. Not yet published.
107. J.R. Barber: *Elasticity*. Second Edition. 2002 ISBN Hb 1-4020-0964-X; Pb 1-4020-0966-6
108. C. Miehe (ed.): *IUTAM Symposium on Computational Mechanics of Solid Materials at Large Strains*. Proceedings of the IUTAM Symposium held in Stuttgart, Germany, 20-24 August 2001. 2003 ISBN 1-4020-1170-9

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

109. P. Ståhle and K.G. Sundin (eds.): *IUTAM Symposium on Field Analyses for Determination of Material Parameters – Experimental and Numerical Aspects.* Proceedings of the IUTAM Symposium held in Abisko National Park, Kiruna, Sweden, July 31 – August 4, 2000. 2003 ISBN 1-4020-1283-7
110. N. Sri Namachchivaya and Y.K. Lin (eds.): *IUTAM Symposium on Nonlinear Stochastic Dynamics.* Proceedings of the IUTAM Symposium held in Monticello, IL, USA, 26 – 30 August, 2000. 2003 ISBN 1-4020-1471-6
111. H. Sobieckzky (ed.): *IUTAM Symposium Transsonicum IV*. Proceedings of the IUTAM Symposium held in Göttingen, Germany, 2–6 September 2002, 2003 ISBN 1-4020-1608-5
112. J.-C. Samin and P. Fisette: *Symbolic Modeling of Multibody Systems*. 2003 ISBN 1-4020-1629-8
113. A.B. Movchan (ed.): *IUTAM Symposium on Asymptotics, Singularities and Homogenisation in Problems of Mechanics.* Proceedings of the IUTAM Symposium held in Liverpool, United Kingdom, 8-11 July 2002. 2003 ISBN 1-4020-1780-4
114. S. Ahzi, M. Cherkaoui, M.A. Khaleel, H.M. Zbib, M.A. Zikry and B. LaMatina (eds.): *IUTAM Symposium on Multiscale Modeling and Characterization of Elastic-Inelastic Behavior of Engineering Materials.* Proceedings of the IUTAM Symposium held in Marrakech, Morocco, 20-25 October 2002. 2004 ISBN 1-4020-1861-4
115. H. Kitagawa and Y. Shibutani (eds.): *IUTAM Symposium on Mesoscopic Dynamics of Fracture Process and Materials Strength.* Proceedings of the IUTAM Symposium held in Osaka, Japan, 6-11 July 2003. Volume in celebration of Professor Kitagawa's retirement. 2004 ISBN 1-4020-2037-6
116. E.H. Dowell, R.L. Clark, D. Cox, H.C. Curtiss, Jr., K.C. Hall, D.A. Peters, R.H. Scanlan, E. Simiu, F. Sisto and D. Tang: *A Modern Course in Aeroelasticity*. 4th Edition, 2004 ISBN 1-4020-2039-2
117. T. Burczyński and A. Osyczka (eds.): *IUTAM Symposium on Evolutionary Methods in Mechanics.* Proceedings of the IUTAM Symposium held in Cracow, Poland, 24-27 September 2002. 2004 ISBN 1-4020-2266-2
118. D. Ieşan: *Thermoelastic Models of Continua*. 2004 ISBN 1-4020-2309-X
119. G.M.L. Gladwell: *Inverse Problems in Vibration*. Second Edition. 2004 ISBN 1-4020-2670-6
120. J.R. Vinson: *Plate and Panel Structures of Isotropic, Composite and Piezoelectric Materials, Including Sandwich Construction*. 2005 ISBN 1-4020-3110-6
121. *Forthcoming*
122. G. Rega and F. Vestroni (eds.): *IUTAM Symposium on Chaotic Dynamics and Control of Systems and Processes in Mechanics*. Proceedings of the IUTAM Symposium held in Rome, Italy, 8–13 June 2003. 2005 ISBN 1-4020-3267-6
123. E.E. Gdoutos: *Fracture Mechanics. An Introduction. 2nd edition.* 2005 ISBN 1-4020-3267-6
124. M.D. Gilchrist (ed.): *IUTAM Symposium on Impact Biomechanics from Fundamental Insights to Applications*. 2005 ISBN 1-4020-3795-3
125. J.M. Huyghe, P.A.C. Raats and S. C. Cowin (eds.): *IUTAM Symposium on Physicochemical and Electromechanical Interactions in Porous Media*. 2005 ISBN 1-4020-3864-X
126. H. Ding, W. Chen and L. Zhang: *Elasticity of Transversely Isotropic Materials*. 2005 ISBN 1-4020-4033-4
127. W. Yang (ed): *IUTAM Symposium on Mechanics and Reliability of Actuating Materials.* Proceedings of the IUTAM Symposium held in Beijing, China, 1–3 September 2004. 2005 ISBN 1-4020-4131-6

Mechanics

SOLID MECHANICS AND ITS APPLICATIONS

Series Editor: G.M.L. Gladwell

128. J.-P. Merlet: *Parallel Robots.* 2006 ISBN 1-4020-4132-2
129. G.E.A. Meier and K.R. Sreenivasan (eds.): *IUTAM Symposium on One Hundred Years of Boundary Layer Research.* Proceedings of the IUTAM Symposium held at DLR-Göttingen, Germany, August 12–14, 2004. 2006 ISBN 1-4020-4149-7
130. H. Ulbrich and W. Günthner (eds.): *IUTAM Symposium on Vibration Control of Nonlinear Mechanisms and Structures.* 2006 ISBN 1-4020-4160-8
131. L. Librescu and O. Song: *Thin-Walled Composite Beams.* Theory and Application. 2006 ISBN 1-4020-3457-1
132. G. Ben-Dor, A. Dubinsky and T. Elperin: *Applied High-Speed Plate Penetration Dynamics*. 2006 ISBN 1-4020-3452-0
133. X. Markenscoff and A. Gupta (eds.): *Collected Works of J. D. Eshelby*. Mechanics of Defects and Inhomogeneities. 2006 ISBN 1-4020-4416-X
134. R.W. Snidle and H.P. Evans (eds.): *IUTAM Symposium on Elastohydrodynamics and Microelastohydrodynamics*. Proceedings of the IUTAM Symposium held in Cardiff, UK, 1–3 September, 2004. 2006 ISBN 1-4020-4532-8
135. T. Sadowski (ed.): *IUTAM Symposium on Multiscale Modelling of Damage and Fracture Processes in Composite Materials*. Proceedings of the IUTAM Symposium held in Kazimierz Dolny, Poland, 23–27 May 2005. 2006 ISBN 1-4020-4565-4
136. A. Preumont: *Mechatronics*. Dynamics of Electromechanical and Piezoelectric Systems. 2006 ISBN 1-4020-4695-2
137. M.P. Bendsøe, N. Olhoff and O. Sigmund (eds.): *IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials*. Status and Perspectives. 2006 ISBN 1-4020-4729-0
138. A. Klarbring: *Models of Mechanics*. 2006 ISBN 1-4020-4834-3
139. H.D. Bui: *Fracture Mechanics*. Inverse Problems and Solutions. 2006 ISBN 1-4020-4836-X
140. M. Pandey, W.-C. Xie and L. Xu (eds.): *Advances in Engineering Structures, Mechanics and Construction*. Proceedings of an International Conference on Advances in Engineering Structures, Mechanics & Construction, held in Waterloo, Ontario, Canada, May 14–17, 2006. 2006 ISBN 1-4020-4890-4
141. G.Q. Zhang, W.D. van Driel and X. J. Fan: *Mechanics of Microelectronics*. 2006 ISBN 1-4020-4934-X
142. Q.P. Sun and P. Tong (eds.): *IUTAM Symposium on Size Effects on Material and Structural Behavior at Micron- and Nano-Scales*. Proceedings of the IUTAM Symposium held in Hong Kong, China, 31 May–4 June, 2004. 2006 ISBN 1-4020-4945-5
143. A.P. Mouritz and A.G. Gibson: *Fire Properties of Polymer Composite Materials*. 2006 ISBN 1-4020-5355-X
144. Y.L. Bai, Q.S. Zheng and Y.G. Wei (eds.): *IUTAM Symposium on Mechanical Behavior and Micro-Mechanics of Nanostructured Materials*. Proceedings of the IUTAM Symposium held in Beijing, China, 27–30 June 2005. 2007 ISBN 1-4020-5623-0
145. L.P. Pook: *Metal Fatigue*. What It Is, Why It Matters. 2007 ISBN 1-4020-5596-6
146. H.I. Ling, L. Callisto, D. Leshchinsky, J. Koseki (eds.): *Geomechanics: Laboratory Testing, Modeling and Applications.* A Collection of Papers of the Geotechnical Symposium in Rome, March 16-17, 2006 ISBN 978-1-4020-6145-5
147. A.S. Kravchuk and P.J. Neittaanmäki: *Variational and Quasi-Variational Inequalities in Mechanics.* 2007. ISBN 978-1-4020-6376-3